Correction of the exercises from the book *A Wavelet Tour of Signal Processing*

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Abstract

These corrections refer to the 3rd edition of the book *A Wavelet Tour of Signal Processing – The Sparse Way* by Stéphane Mallat, published in December 2008 by Elsevier. If you find mistakes or imprecisions in these corrections, please send an email to Gabriel Peyré (gabriel.peyre@ceremade.dauphine.fr). More information about the book, including how to order it, numerical simulations, and much more, can be find online at wavelet-tour.com.

1 Chapter 2

Exercise 2.1. For all t, the function $\omega \mapsto e^{-i\omega t} f(t)$ is continuous. If $f \in \mathrm{L}^1(\mathbb{R})$, then for all ω, $|e^{-i\omega t} f(t)| \leqslant |f(t)|$ which is integrable. One can thus apply the theorem of continuity under the integral sign $\int$ which proves that $\hat{f}$ is continuous.

If $\hat{f} \in \mathrm{L}^1(\mathbb{R})$, using the inverse Fourier formula (2.8) and a similar argument, one proves that f is continuous.

Exercise 2.2. If $\int |h| = +\infty$, for all $A > 0$ there exists $B > 0$ such that $\int_{-B}^{B} |h| > A$. Taking $f(x) = 1_{[-A,A]} \operatorname{sign}(h(-x))$ which is integrable and bounded by 1 shows that

$$f \star h(0) = \int_{-B}^{B} \operatorname{sign}(h(t)) h(t) \mathrm{d}t > A.$$

This shows that the operator $f \mapsto f \star h$ is not bounded on L^∞, and thus the filter h is unstable.

Exercise 2.3. Let $f_u(t) = f(t-u)$, by change of variable $t - u \to t$, one gets

$$\hat{f}_u(\omega) = \int f(t-u) e^{-i\omega t} \mathrm{d}t = \int f(t) e^{-i\omega(t+u)} \mathrm{d}t = e^{-i\omega u} \hat{f}(\omega).$$

Let $f_s(t) = f(t/s)$, with $s > 0$, by change of variable $t/s \mapsto t$, one get

$$\hat{f}_s(\omega) = \int f(t/s) e^{-i\omega t} \mathrm{d}t = \int f(t) e^{-i\omega st} |s| \mathrm{d}t = |s| \hat{f}(s\omega).$$

Let f by C^1 and $g = f'$, the by integration by parts, since $f(t) \to 0$ where $|t| \to +\infty$,

$$\hat{g}(\omega) = \int f'(t)e^{-i\omega t}\mathrm{d}t = -\int f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t = (i\omega)\hat{f}(\omega).$$

Exercise 2.4. One has

$$f_r(t) = \mathrm{Re}[f(t)] = [f(t) + f^*(t)]/2 \quad \text{and} \quad f_i(t) = \mathrm{Ima}[f(t)] = [f(t) - f^*(t)]/2$$

so that

$$\begin{aligned}\hat{f}_r(\omega) &= \int \frac{f(t) + f^*(t)}{2} e^{-i\omega t}\mathrm{d}t = \hat{f}(\omega)/2 + \mathrm{Conj}\left(\int f(t)e^{i\omega t}\mathrm{d}t\right)/2 \\ &= [\hat{f}(\omega) + \hat{f}^*(-\omega)]/2,\end{aligned}$$

where $\mathrm{Conj}(a) = a^*$ is the complex conjugate. The same computation leads to

$$\hat{f}_i(\omega) = [\hat{f}(\omega) - \hat{f}^*(-\omega)]/2.$$

Exercise 2.5. One has

$$\hat{f}(0) = \int f(t)\mathrm{d}t = 0.$$

If $f \in \mathrm{L}^1(\mathbb{R})$, one can apply the theorem of derivation under the integral sign $\int$ and get

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{f}(\omega) = \int -itf(t)e^{-i\omega t}\mathrm{d}t \quad \Longrightarrow \quad \hat{f}'(0) = -i\int tf(t)\mathrm{d}t = 0.$$

Exercise 2.6. If $f = 1_{[-\pi,\pi]}$ then one can verify that

$$\hat{f}(\omega) = \frac{2\sin(\pi\omega)}{\omega}.$$

It result that

$$\int \frac{\sin(\pi\omega)}{\pi\omega} = \frac{1}{2\pi}\int \hat{f}(\omega)\mathrm{d}\omega = f(0) = 1.$$

If $g = 1_{[-1,1]}$ then $\hat{g}(\omega)/2 = \sin(\omega)/\omega$. The inverse Fourier transform of $\hat{g}(\omega)^3$ is $g \star g \star g(t)$ so

$$\int \frac{\sin^3(\omega)}{\omega^3}\mathrm{d}\omega = \frac{1}{8}\int \hat{g}(\omega)^3\mathrm{d}\omega = \frac{2\pi}{8} g \star g \star g(0) = \frac{3\pi}{4},$$

where we used the fact that

$$g \star g \star g(0) = \int_{-1}^{1} h(t)\mathrm{d}t = 3$$

where h is a piecewise linear hat function with $h(0) = 2$.

Exercise 2.7. Writing $u = a - ib$, and differentiating under the integral sign $\int$, one has

$$f'(\omega) = \int -ite^{-ut^2}e^{-i\omega t}\mathrm{d}t.$$

By integration by parts, one gets an ordinary differential equation

$$f'(\omega) = \frac{-\omega}{2u}\hat{f}(\omega)$$

whose solution is

$$f(\omega) = K e^{-\frac{\omega^2}{4u}}$$

for some constant $K = \hat{f}(0)$. Using a switch from Euclidean coordinates to polar coordinates $(x, y) \to (r, \theta)$ which satisfies $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$, one gets

$$\begin{aligned} K^2 &= \int e^{-ux^2}\mathrm{d}x \int e^{-uy^2}\mathrm{d}y = \iint e^{-u(x^2+y^2)}\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\int_0^{+\infty} e^{-ur^2} r\mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^{+\infty} re^{-ur^2}\mathrm{d}r = \frac{\pi}{u}, \end{aligned}$$

which gives the result.

Exercise 2.8. If f is $\mathbf{C}^1$ with a compact support, with an integration by parts we get

$$\hat{f}(\omega) = \frac{1}{i\omega}\int f'(t)e^{-i\omega t}\mathrm{d}t$$

so that

$$|\hat{f}(\omega)| \leqslant \frac{C}{\omega} \quad \text{with} \quad C = \int |f'(t)|\mathrm{d}t < +\infty,$$

which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Let $f \in \mathbf{L}^1(\mathbb{R})$ and $\varepsilon > 0$. Since $\mathbf{C}^1$ functions are dense in $\mathbf{L^1}(\mathbb{R})$, one can find g such that $\int |f - g| \leqslant \varepsilon/2$. Since $\hat{g}(\omega) \to 0$ when $|\omega| \to +\infty$, there exists A such that $|\hat{g}(\omega)| \leqslant \varepsilon/2$ when $|\omega| > A$. Moreover, the Fourier integral definition implies that

$$|\hat{f}(\omega) - \hat{g}(\omega)| \leqslant \int |f(t) - g(t)|\, dt$$

so for all $|\omega| > A$ we have $|\hat{f}(\omega)| \leqslant \varepsilon$ which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Exercise 2.9. a) For $f_0(t) = 1_{[0,+\infty)}(t)e^{pt}$, we get

$$\hat{f}_0(\omega) = \int_0^{+\infty} e^{(p-i\omega)t}\mathrm{d}t = \frac{1}{i\omega - p}.$$

For $f_n(t) = t^n 1_{[0,+\infty)}(t)e^{pt}$, an integration by parts gives

$$\hat{f}_n(\omega) = \int_0^{+\infty} t^n e^{(p-i\omega)t}\mathrm{d}t = \frac{n}{i\omega - p}\hat{f}_{n-1}(\omega),$$

so that

$$\hat{f}_n(\omega) = \frac{n!}{(i\omega - p)^n}.$$

b) Computing the Fourier transform on both sides of the differential equation gives

$$g = f \star h \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k (i\omega)^k}{\sum_{k=0}^{M} b_k (i\omega)^k}.$$

We denote by $\{p_k\}_{k=0}^{L}$ the poles of the polynomial $\sum_{k=0}^{M} b_k z^k$, with multiplicity n_k. If $K < M$, one can decompose the rational fraction into

$$\hat{h}(\omega) = \sum_{k=0}^{L} \frac{Q_k(i\omega)}{(i\omega - p_k)^{n_k}}$$

where each Q_k is a polynomial of degree strictly smaller than n_k. It results that $h(t)$ is a sum of derivatives up to a degree strictly smaller than n_k of the inverse Fourier transform of

$$\hat{f}_{p_k,n_k}(\omega) = \frac{1}{(i\omega - p_k)^{n_k}}$$

which is

$$f_{p_k,n_k}(t) = \frac{1}{n_k!} t^{n_k} 1_{[0,+\infty)}(t) e^{p_k t}.$$

Each filter f_{p_k,n_k} is causal, stable and n_k times differentiable. It results that that h is causal and stable.

If, there exists l with $\mathrm{Re}(p_l) = 0$ then for the frequency $\omega = -ip_l$ we have $|\hat{h}(\omega)| = +\infty$ so h can not be stable.

If, there exists l with $\mathrm{Re}(p_l) > 0$ then by observing that $\hat{f}_{p_l,n_l}(-\omega) = (-1)^{n_l}(i\omega + p_l)^{-n_l}$ and by applying the result in a) we get

$$f_{p_l,n_{k_l}}(t) = \frac{1}{n_l!} t^{n_l} 1_{(-\infty,0]}(t) e^{-p_l t}$$

which is anticausal. We thus derive that h is not causal.

c) Denoting $\alpha = e^{i\pi/3}$, one can write

$$|\hat{h}(\omega)|^2 = \frac{1}{1 - (i\omega/\omega_0)^6}$$

with

$$1/\hat{h}(\omega) = (i\omega/\omega_0 + 1)(i\omega/\omega_0 + \alpha)(i\omega/\omega_0 + \alpha^*) = P(i\omega).$$

Since the zeros of $P(z)$ have all a strictly negative real part, h is stable and causal. To compute $h(t)$ we decompose

$$\hat{h}(\omega) = \frac{a_1}{i\omega/\omega_0 + 1} + \frac{a_2}{i\omega/\omega_0 + \alpha} + \frac{a_3}{i\omega/\omega_0 + \alpha^*},$$

we compute a_1, a_2 and a_3 and by applying the result in (a) we derive that

$$\hat{h}(t) = \omega_0 (a_1\ 1_{[0,+\infty)}(t)\, e^{-t\omega_0} + a_2\ 1_{[0,+\infty)}(t)\, e^{-t\alpha\omega_0} + a_3\ 1_{[0,+\infty)}(t)\, e^{-t\alpha^*\omega_0})\ .$$

Exercise 2.10. For $a > 0$ and $u > 0$ and g a Gaussian function, define

$$f_{a,u}(t) = e^{iat} g(t - u) + e^{-iat} g(t + u).$$

We verify that $\sigma_\omega(f_{a,u})$ increases proportionally to u. Its Fourier transform is

$$\hat{f}_{a,u}(\omega) = e^{-iu\omega} \hat{g}(\omega - a) + e^{iu\omega} \hat{g}(\omega + a)$$

so $\sigma_\omega(f_{a,u})$ increases proportionally to a. For a and u sufficiently large we get the the result.

Exercise 2.11. Since $f(t) \geqslant 0$

$$|\hat{f}(\omega)| = |\int f(t)\, e^{-i\omega t}\, dt| \leqslant \int f(t)\, dt = \hat{f}(0)\ .$$

Exercise 2.12. a) Denoting $u(t) = |\sin(t)|$, one has $g(t) = a(t) u(\omega_0 t)$ so that

$$\hat{g}(\omega) = \frac{1}{2\pi} \hat{a}(\omega) \star \hat{u}(\omega/\omega_0)$$

where $\hat{u}(\omega)$ is a distribution

$$\hat{u}(\omega) = \sum_n c_n \delta(\omega - n)$$

and c_n is the Fourier coefficient

$$c_n = \int_{-\pi}^{\pi} |\sin(t)| e^{-int} \mathrm{d}t = -\int_{-\pi}^{0} \sin(t) e^{-int} \mathrm{d}t + \int_{0}^{\pi} \sin(t) e^{-int} \mathrm{d}t.$$

The change of variable $t \to t + \pi$ in the first integral shows that $c_{2k+1} = 0$ and for $n = 2k$,

$$c_{2k} = 2 \int_0^{\pi} \sin(t) e^{-i2kt} \mathrm{d}t = \frac{4}{1 - 4k^2}.$$

One thus has

$$\hat{u}(\omega) = \frac{1}{2\pi} \sum_n c_n \hat{a}(\omega - n\omega_0) = \frac{2}{\pi} \sum_k \frac{\hat{a}(\omega - 2k\omega_0)}{1 - 4k^2}.$$

b) If $\hat{a}(\omega) = 0$ for $|\omega| > \omega_0$, then h defined by $\hat{h}(\omega) = \frac{\pi}{2} 1_{[-\omega_0, \omega_0]}$ guarantees that $\hat{g}\hat{h} = \hat{a}$ and hence $a = g \star h$.

Exercise 2.13. One has

$$\hat{g}(\omega) = \frac{1}{2} \sum_n \hat{f}_n(\omega) \star [\delta(\omega - 2n\omega_0) + \delta(\omega + 2n\omega_0)] = \frac{1}{2} \sum_n [\hat{f}_n(\omega - 2n\omega_0) + \hat{f}_n(\omega + 2n\omega_0)].$$

Each $\hat{f}_n(\omega \pm 2n\omega_0)$ is supported in $[(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]$, and thus $\hat{g}$ is supported in $[-2N\omega_0, 2N\omega_0]$.

Since the intervals $[(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]$ are disjoint, one has

$$\hat{f}_n(\omega \pm 2n\omega_0) = 2\hat{g}(\omega) 1_{[(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]}(\omega).$$

The change of variable $\omega \pm 2n\omega_0 \to \omega$ and summing for n and $-n$ gives

$$\hat{f}_n(\omega) = [\hat{g}(\omega - 2n\omega_0) + \hat{g}(\omega + 2n\omega_0)]\hat{h}(\omega),$$

where $\hat{h}(\omega) = 1_{[-\omega_0, \omega_0]}(\omega)$. Denoting $g_n(t) = 2g(t)\cos(2n\omega_0 t)$, one sees that f_n is recovered as

$$f_n = g_n \star h.$$

Exercise 2.14. The function $\phi(t) = \sin(\pi t)/(\pi t)$ is monotone on $[-3/2, 0]$ and $[0, 3/2]$ on which is variation is $1 + \frac{2}{3\pi}$. For each $k \in \mathbb{N}^*$, it is also monotone on each interval $[k + 1/2, k + 3/2]$ on which the variation is $\frac{1}{\pi}[(k + 1/2)^{-1} + (k + 3/2)^{-1}]$. One thus has

$$\|\phi\|_V = 2(1 + \frac{2}{3\pi}) + \frac{2}{\pi} \sum_{k \geqslant 1} [(k + 1/2)^{-1} + (k + 3/2)^{-1}] = +\infty.$$

For $\phi = \lambda 1_{[a,b]}$, $|\phi'| = \lambda\delta_a + \lambda\delta_b$ and hence $\|\phi\|_V = 2\lambda$.

Exercise 2.16. Let

$$f(x) = 1_{[0,1]^2}(x_1, x_2) = f_0(x_1) f_0(x_2) \quad \text{where} \quad f_0(x_1) = 1_{[0,1]}(x_1).$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \frac{(e^{i\omega_1} - 1)(e^{i\omega_2} - 1)}{\omega_1\omega_2}.$$

Let

$$f(x) = e^{-x_1^2 - x_2^2} = f_0(x_1)f_0(x_2) \quad \text{where} \quad f_0(x_1) = e^{-x_1^2}.$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \pi e^{-(\omega_1^2 + \omega_2^2)/4}.$$

Exercise 2.17. If $|t| > 1$, the ray $\Delta_{t,\theta}$ does not intersect the unit disc, and thus $p_\theta(t) = 0$. For $|t| < 1$, the Radon transform is computed as the length of a cross section of a disc

$$p_\theta(t) = 2\sqrt{1 - t^2}.$$

Exercise 2.18. We prove that the Gibbs oscillation amplitude is independent of the angle θ and is equal to a one-dimensional Gibbs oscillation. Let us decompose $f(x)$ into a continuous part $f_0(x)$ and a discontinuity of constant amplitude A:

$$f(x) = f_0(x) + A\, u(\cos(\theta)x_1 + \sin(\theta)x_2)$$

where $u(t) = 1_{[0,+\infty)}(t)$ is the one-dimensional Heaviside function. The filter satisfies $h_\xi(x_1, x_2) = g_\xi(x_1)\, g_\xi(x_2)$ with $g_\xi(t) = \sin(\xi t)/(\pi t)$. The Gibbs phenomena is produced by the discontinuity corresponding to the Heaviside function so we can consider that $f_0 = 0$. Let us suppose that $|\theta| \leqslant \pi/4$, with no loss of generality. We first prove that

$$f \star h_\xi(x) = f \star g_\xi(x) \tag{1}$$

where $\hat{g}_\xi(\omega_1, \omega_2) = 1_{[-\xi,\xi]}(\omega_2)$. Indeed $f(x)$ is constant along any line of angle θ, one can thus verify that its Fourier transform has a support located on the line in the Fourier plane, of angle $\theta + \pi/2$ which goes through 0. It results that $\hat{f}(\omega)\hat{h}_\xi(\omega) = \hat{f}(\omega)\hat{g}_\xi(\omega)$ because the filtering limits the support of $\hat{f}$ to $|\omega_2| \leqslant \xi$. But $g_\xi(x_1, x_2) = \delta(x_1)\, \sin(\xi x_2)/(\pi x_2)$. The convolution (1) is thus a one-dimensional convolution along the x_2 variable, which is computed in the Gibbs Theorem 2.8. The resulting one-dimensional Gibbs oscillations are of the order of $A \times 0.045$.

2 Chapter 3

Exercise 3.1. One has $\phi_{s,n}(t) = s^{-1/2} 1_{[ns,(n+1)s)}$, which satisfies $\|\phi_{s,n}\| = 1$ and $\langle \phi_{s,n},\, \phi_{s,n'} \rangle = 0$ for $n \neq n'$ because $[ns, (n+1)s)$ and $[n's, (n'+1)s)$ are disjoint. If $f(x) = a_n$ on each interval $[ns, (n+1)s)$, then

$$f(x) = \sum_n a_n 1_{[ns,(n+1)s)} = \sum_n \langle f,\, \phi_{s,n} \rangle \phi_{s,n}$$

So $\{\phi_{s,n}\}_n$ is an orthonormal basis of functions that are piecewise constant on each interval $[ns, (n+1)s)$.

Exercise 3.2. If $\mathrm{Supp}(\hat{f}) \subset [-\pi/s, \pi/s]$, then

$$\hat{f}(\omega) = \hat{f}(\omega) 1_{[-\pi/s,\pi/s]}(\omega) = \frac{1}{s}\hat{f}(\omega)\hat{\phi}_s(\omega)$$

and hence using the Fourier convolution theorem and the fact that ϕ_s is symmetric,

$$f(u) = \frac{1}{s} f \star \phi_s(u) = \frac{1}{s}\langle f(t),\, \phi_s(t-u)\rangle.$$

Exercise 3.3. a) The function f is an interpolation function if and only if

$$\sum_n f(n)\delta(t-n) = \delta(t)$$

in the distribution sense. Using the sampling Theorem 3.1 with $s=1$, one gets equivalently the equality of the Fourier transform

$$\sum_k \hat{f}(\omega + 2k\pi) = 1.$$

b) One has

$$\hat{f}(\omega) = \sum_n h[n]\theta(\omega)e^{-in\omega} = h(\omega)\theta(\omega).$$

and thus

$$\sum_k \hat{f}(\omega+2k\pi) = h(\omega)A(\omega) \quad \text{where} \quad A(\omega) = \sum_k \hat{\theta}(\omega+2k\pi)$$

If for all ω, $A(\omega) \neq 0$, one can set $h(\omega) = 1/A(\omega)$ and $f(\omega) = \theta(\omega)/A(\omega)$. Thus $f \in \mathrm{L}^2(\mathbb{R})$ if it exists $B > 0$ such that $|A(\omega)| > B$, in which case $\|f\| \leqslant \|\theta\|/B$.

Exercise 3.4. Let $A(t) = \sum_n f(t-n)$, then in the sense of distribution

$$\hat{A}(\omega) = \hat{f}(\omega)\sum_n e^{-in\omega} = \hat{f}(\omega)\sum_n \delta(\omega - 2n\pi) = \sum_n \hat{f}(2n\pi)\delta(\omega-2n\pi).$$

Taking the inverse Fourier transform leads to

$$\sum_n f(t-n) = \sum_n \hat{f}(2n\pi)e^{2in\pi\omega}.$$

Exercise 3.5. The orthogonal projection of f on U_s is defined by

$$\forall\, g \in U_s, \quad \langle \tilde{f} - f,\, \tilde{f} - g\rangle = 0.$$

It thus satisfies

$$\forall\, g \in U_s, \quad \|f-g\|^2 = \|f-\tilde{f}\|^2 + \|\tilde{f}-g\|^2 \geqslant \|f-\tilde{f}\|^2$$

and hence $\tilde{f}$ minimizes $\|f-\tilde{f}\|$ subject to $\tilde{f} \in U_s$.

Exercise 3.6. The sufficient condition comes from

$$\|Lf\|_\infty \leqslant \|f\|_\infty \|h\|_1.$$

If $\|h\|_\infty = +\infty$, then for any $A > 0$ it exists B such that $\sum_{|k|<B} |h[k]| > A$. Taking $f[k] = \mathrm{sign}(h[-k])1_{[-B,B]}[-k]$ that satisfies $\|f\|_\infty \leqslant 1$ shows that

$$Lf[0] = \sum_{|k|\leqslant B} |h[k]| > A.$$

The operator $f \mapsto f \star h$ is not bounded on ℓ^∞, so that the filter is unbounded.

Exercise 3.7. One has

$$\hat{g} \star \hat{h}(\omega) = \int_{-\pi}^{\pi} \left(\sum_n h[n] e^{-in\xi} \right) \left(\sum_p g[p] e^{-ip(\omega-\xi)} \right) \mathrm{d}\xi$$
$$= \int_{-\pi}^{\pi} \sum_{n,p} h[n] g[p] e^{-i\xi(n-p)} e^{-i\omega p} \mathrm{d}\xi.$$

Exchanging signs $\int$ and $\sum$, and using the fact that

$$\int_{-\pi}^{\pi} e^{-i\xi(n-p)} \mathrm{d}\xi = \delta[n-p]$$

shows that

$$\hat{g} \star \hat{h}(\omega) = \sum_n h[n] g[n] e^{-in\omega} = \hat{f}(\omega).$$

Exercise 3.8. Let $e_k[n] = e^{\frac{2i\pi}{N} kn} = \omega^{kn}$ where $\omega = e^{\frac{2i\pi}{N}}$. If $k \neq k'$, one has a geometrical sum

$$\langle e_k,\, e_{k'} \rangle = \sum_n \omega^{kn} \omega^{-k'n} = \sum_n (\omega^{k-k'})^n = \frac{1 - \omega^{N(k-k')}}{1 - \omega^{k-k'}} = 0$$

because $\omega^N = 1$. Since $\|e_k\| = \sqrt{N}$, the family $\{e_k/\sqrt{N}\}_k$ is an orthonormal basis of $\mathbb{C}^N$.

Exercise 3.9. Denoting

$$f_d(t) = \sum_k f(ks) \delta(t - ks),$$

one has using Theorem 3.1

$$\hat{f}_d(\omega) = \frac{1}{s} \sum_k \hat{f}(\omega - 2k\pi/s).$$

Since $\hat{f}$ is supported in $I_n = [-(n+1)\pi/s, -n\pi/s] \cup [n\pi/s, (n+1)\pi/s]$, and the intervals $I_n + 2k\pi$ are disjoint, one has

$$\hat{f}(\omega) = \hat{f}_d(\omega) \hat{\phi}_s(\omega) \quad \text{where} \quad \phi_s(\omega) = s 1_{I_n}(\omega),$$

which corresponds to the reconstruction formula

$$f(t) = \sum_n f(ns) \phi_s(t - ns)$$

with the kernel obtained by inverse Fourier transform formula

$$\phi_s(t) = s \int_{-(n+1)\pi/s}^{-n\pi/s} e^{i\omega t} \mathrm{d}\omega + s \int_{n\pi/s}^{(n+1)\pi/s} e^{i\omega t} \mathrm{d}\omega$$
$$= \frac{1}{\pi\omega/s} \left[\sin((n+1)\pi/s) - \sin(n\pi/s) \right]$$

Exercise 3.10. a) One has $\tilde{f}(ns) = f \star \phi_s$ where $\phi_s = 1_{[-s/2, s/2]}$.

b) Since $\hat{\tilde{f}}(\omega) = \hat{f}(\omega) \hat{\phi}_s(\omega)$, $\mathrm{supp}(\hat{\tilde{f}}) \subset [-\pi/s, \pi/s]$, and hence $\tilde{f}$ is recovered using Shannon interpolation formula.

c) One has

$$\hat{\tilde{f}}(\omega) = \hat{f}(\omega)\frac{\sin(\omega s/2)}{\omega s/2}$$

and thus

$$f = \tilde{f} \star \psi_s \quad \text{with} \quad \hat{\psi}_s(\omega) = \frac{\omega s/2}{\sin(\omega s/2)}.$$

d) For $\omega \in [-\pi/s, \pi/s]$, one has

$$\hat{\phi}_s(\omega) > 2/\pi$$

and hence $\|f\| \leqslant \pi/2\|\tilde{f}\|$ which shows that the reconstruction is stable.

Exercise 3.11. a) ϕ is supported in $[-1,1]$, on $t \in [-1,0], \phi(t) = t+1$, on $t \in [0,1], \phi(t) = 1-t$.
b) If $f(t)$ is linear on each interval $[n, n+1]$, then

$$f(t) = \sum_n f(n)\phi(t-n).$$

One has, using (7.21),

$$\sum_k |\hat{\phi}(\omega - 2k\pi)|^2 = \sum_k \frac{\sin^4(\omega/2)}{(\omega/2 + k\pi)^2} = \frac{1}{3}(1 + 2\cos^2(\omega/2)) \geqslant \frac{1}{3},$$

so using Theorem 3.4, $\{\phi(t-n)\}_n$ is a Riesz basis of the space of piecewise linear functions on each interval $[n, n+1]$.
c) The dual basis satisfies

$$\hat{\tilde{\phi}}(\omega) = \frac{3\sin^2(\omega/2)}{(\omega/2)^2(1 + 2\cos^2(\omega/2))} = \frac{\hat{h}(\omega)}{-\omega^2}$$

where $\hat{h}(om) = -12\sin^2(\omega/2)/(1+2\cos^2(\omega/2))$ is the Fourier series of a discrete filter. It results that $\phi(t)$ is obtained by integrating twice $h(t) = \sum_n h[n]\delta(t-n)$. The Fourier series $\hat{h}(\omega)$ is a rational fraction of $e^{-i\omega}$ which is not reductible to a polynomial so $h[n]$ has an infinite support, which proves that $\phi(t)$ also has an infinite support.

Exercise 3.12. One has

$$|\hat{f}[k]| = |\sum_n f[n]e^{-\frac{2i\pi}{N}nk}| \leqslant \sum_n |f[n]||e^{-\frac{2i\pi}{N}nk}| = \sum_n |f[n]|.$$

Exercise 3.13. a) Let $h[0] = 1$, $h[-1] = -1$ and $h[n] = 0$ for $n \notin \{-1, 0\}$. Then

$$\|f\|_V = \sum_n |f \circledast h[n]|$$

and

$$\hat{h}[k] = 1 - e^{\frac{2i\pi}{N}k} \quad \Longrightarrow \quad |h[k]| = 2|\sin(k\pi/N)|.$$

b) By applying the result of the exercise 3.12 we have that the Fourier transform $\hat{f}[k]\,\hat{h}[k]$ of $f \circledast h[n$ satisfies

$$|\hat{f}[k]\,\hat{h}[k]| \leqslant \|f\|_V$$

so for $|k| \leqslant N/2$ we verify that

$$|\hat{f}[k]| \leqslant \frac{\|f\|_V}{2|\sin(k\pi/N)|} \leqslant \frac{\|f\|_V}{2|\sin(k\pi/N)|} \leqslant \frac{N\,\|f\|_V}{2\,k}$$

since $\sin(x) \geqslant 2x/\pi$ for $|x| \leqslant \pi/2$.

Exercise 3.14. Since $(-1)^n = e^{-i\pi n}$, one has

$$\hat{g}(\omega) = \sum_n h[n]e^{i\pi n}e^{in\omega} = \hat{h}(\omega+\pi).$$

If h is a low-pass filter, then g is a high-pass filter.

Exercise 3.15. If $g = f \star h$ then $\|g\|_1 \leqslant \|f\|_1\|h\|_1$ and we can exchange the summations order as follow

$$\begin{aligned}\hat{g}(\omega) &= \sum_n e^{-i\omega n}\sum_p f[p]h[n-p] = \sum_p f[p]e^{-i\omega p}\sum_n e^{-i\omega(n-p)}h[n-p] \\ &= \sum_p f[p]e^{-i\omega p}\sum_n e^{-i\omega n}h[n] = \hat{f}(\omega)\hat{h}(\omega)\end{aligned}$$

where the second line is obtained by change of variable $n-p \to p$ in the summation.

Exercise 3.16. a) Since $\hat{h}(\omega)\hat{h}^{-1}(\omega) = \hat{\delta}(\omega) = 1$, one has $\hat{h}^{-1} = 1/\hat{h}$.
b) Up to translation we suppose that h is causal. The filter h^{-1} and h have finite support if and only if $\hat{h}(\omega) = P(e^{-i\omega})$ where $P(z)$ and $z^k/P(z)$ are polynomial for some $k \in \mathbb{N}$. This can only happens if $P(z)$ is a monomial $P(z) = az^p$, so that $h[n] = a\delta[n-p]$.

Exercise 3.17. a) One has

$$|h(\omega)|^2 = \prod_{k=1}^{K} \frac{|a_k^* - e^{-i\omega}|^2}{|1+e^{-i\omega}|^2}$$

and one verifies that $|a_k^* - e^{-i\omega}|^2 = 1 + |a_k|^2 + 2\mathrm{Re}(a_k e^{-i\omega}) = |1+e^{-i\omega}|^2$.
b) $\{h[n-m]\}_m$ is an orthogonal basis if and only if for all m, m'

$$\langle h[n-m],\, h[n-m']\rangle = h \star h[m-m'] = \delta[n-m'].$$

Taking the Fourier transform of this relation leads to $\|h(\omega)\|^2 = 1$ for all ω.

Exercise 3.18. a) For $h_0[n] = a^n 1_{[0,+\infty)}[n]$, one has the following geometrical sum

$$\hat{h}_0(\omega) = \sum_n (ae^{-i\omega})^n = \frac{1}{1-ae^{-i\omega}}.$$

Let $h_p(\omega) = (1-ae^{-i\omega})^{-p}$. Observe that

$$h_p'(\omega) = \frac{-paie^{-i\omega}}{(1-ae^{-i\omega})^{p+1}} = \sum_n h_p[n](-in)e^{-in\omega}\ .$$

The inverse Fourier transform of $h_{p+1}(\omega) = (1-ae^{-i\omega})^{-p-1}$ thus satisfies

$$h_{p+1}[n] = \frac{(n+1)\,h_p[n+1]}{a\,p}\ .$$

Iterating on this relation with $h_1[n] = a^n 1_{[0,+\infty)}[n]$ gives

$$h_p[n] = \frac{h_1[n+p-1]\,,\prod_{k=1}^{p-1}(n+k)}{a^{p-1}\,(p-1)!}\ .$$

b) Taking the Fourier transform of the recursion formula shows that

$$\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega) \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k e^{-ik\omega}}{\sum_{k=0}^{M} b_k e^{-ik\omega}}.$$

c) Let a_k be the roots of the polynomial $\sum_{k=0}^{M} b_k z^{-k}$ with multiplicity p_k. Then

$$\hat{h}(\omega) = P(e^{-i\omega}) \prod_k (1 - a_k e^{-i\omega})^{-p_k},$$

where P is a polynomial. If one has $|a_k| < 1$ for all k, then using question a), each $(1 - a_k e^{-i\omega})^{-p_k}$ is the Fourier transform of a stable causal filter, and so is h.

If there exists l such that $|a_l| = 1$, then it can be written $a_l = e^{i\alpha}$ so $|\hat{h}(\alpha)| = +\infty$ and the filter can therefore not be stable.

If, there exists l with $|a_l| > 1$ then let $h_l(\omega) = (1 - a_l e^{-i\omega})^{-p_l}$. Observe that

$$h_l(-\omega) = (1 - a_l e^{i\omega})^{-p_l} = a_l^{-p_l} e^{-i\omega p_l} (a_l^{-1} e^{-i\omega} - 1)^{-p_1}\ .$$

Its inverse Fourier transform is $h_l[-n]$ and with question (a) we verify that it is not a causal filter. To prove that h is then not a causal filter, one can decompose

$$\hat{h}(\omega) = \sum_{k=0}^{L} \frac{Q_k(e^{-i\omega})}{(1 - a_k e^{-i\omega})^{n_k}}$$

where each Q_k is a polynomial of degree strictly smaller than n_k and observe that the component for $k = l$ is not causal so h can not be causal.

Exercise 3.19. One has, using the inverse Fourier transform,

$$\begin{aligned}
\tilde{f}[2n] &= \frac{1}{2N} \sum_{k=0}^{2N-1} \hat{\tilde{f}}[k] e^{\frac{2i\pi}{2N} k2n} \\
&= \frac{1}{N} \sum_{k=0}^{N/2-1} \hat{f}[k] e^{\frac{2i\pi}{N} kn} + \frac{1}{N} \sum_{k=3N/2+1}^{2N-1} \hat{f}[k] e^{\frac{2i\pi}{N} kn} + \frac{1}{N} \hat{f}[N/2] e^{\frac{2i\pi}{N} N/2n},
\end{aligned}$$

and one thus has

$$\tilde{f}[2n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] e^{\frac{2i\pi}{N} kn} = f[n].$$

Exercise 3.20. a) One has

$$\hat{x}(\omega) = \sum_n y[Mn] e^{-i\omega n} = \sum_n (y \cdot h)[n] e^{-i\omega n/M} = \frac{1}{2\pi} \hat{y} \star \hat{h}(\omega/M)$$

where we have use the convolution result of Exercise 3.7 and where

$$h[n] = \sum_k \delta[n - kM]$$

which satisfies, in the sense of distributions

$$\hat{h}(\omega) = \frac{2\pi}{M} \sum_k \delta(\omega - 2\pi k/M).$$

This shows that

$$\hat{x}(\omega) = \frac{1}{M} \int_{-\pi}^{\pi} y(\xi) \sum_k \delta(\omega/M - \xi - 2\pi k/M) \mathrm{d}\xi = \frac{1}{M} \sum_{k=0}^{M-1} \hat{y}((\omega - 2k\pi)/M).$$

b) If for $\omega \in [-\pi, \pi]$, $\hat{y}(\omega) = 0$ for $\omega \notin [-\pi/M, \pi/M]$, then

$$\hat{y}(\omega) = \hat{x}(M\omega)\hat{\phi}_M(\omega) \quad \text{where} \quad \hat{\phi}_M(\omega) = M 1_{[-\pi/M, \pi/M]}(\omega).$$

Let $(x \uparrow M)[Mn] = x[n]$ and $(x \uparrow M)[k] = 0$ if $k \mod M \neq 0$ be the up-sampled signal. The Fourier transform of $x \uparrow M$ is $\hat{x}(M\omega)$, so that

$$y = (x \uparrow M) \star \phi_m \quad \text{where} \quad \phi_M[n] = \frac{\sin(n\pi/M)}{n\pi/M}.$$

Exercise 3.21. a) One has, using the change of variable $r = n - pN$,

$$\hat{f}_p[k] = \sum_{p \in \mathbb{Z}} \sum_{n=0}^{N-1} f_d[n - pN] e^{-\frac{2i\pi}{N} kn} = \sum_{r \in \mathbb{Z}} f_d[r] e^{-\frac{2ik\pi}{N} r} = \hat{f}_d\left(\frac{2k\pi}{N}\right).$$

Using the sampling Theorem 3.1, one gets

$$\hat{f}_p[k] = \hat{f}_d\left(\frac{2k\pi}{N}\right) = \frac{1}{s} \sum_\ell \hat{f}\left(\frac{2k\pi}{Ns} - \frac{2\ell\pi}{s}\right).$$

b) In order to have

$$s\hat{f}_p[k] \approx \hat{f}\left(\frac{2k\pi}{Ns}\right),$$

one needs that $\omega_0 \ll \pi/s$. To be able to interpolate without too much aliasing $\hat{f}$ from the values $\hat{f}\left(\frac{2k\pi}{Ns}\right)$, one needs that $t_0 \ll Ns$. Since a function cannot be compactly supported in both time and space, no exact interpolation formula is possible.
c) The Fourier transform $\hat{f}(\omega)$ is proportional to the convolution of the indicator of $[-\pi, \pi]$ convolved with itself 4 times. Its support is thus $[-4\pi, 4\pi]$.

Exercise 3.22. a) One has

$$\hat{f}[\ell] = \frac{N}{2} \left(\delta[\ell - k] + \delta[\ell + k]\right).$$

So

$$f_a[n] = e^{\frac{2i\pi}{N} kn}.$$

b) One has $g[n] = (f[n] + f^*[n])/2$, and using a change of variable $n \to -n$ in the summation gives

$$\hat{g}[k] = \hat{f}[k]/2 + \sum_n f^*[n] e^{-\frac{2i\pi}{N} kn} = \hat{f}[k]/2 + \hat{f}[-k]^*/2.$$

c) The definition of $\hat{f}_a$ shows that $g = \text{Re}(f_a)$ satisfies

$$\hat{g}[k] = (\hat{f}_a[k] + \hat{f}_a[-k]^*)/2 = \hat{f}[k],$$

and hence $g = f$.

Exercise 3.23. One has

$$\begin{aligned}\langle e_{k_1}[n_1]e_{k_2}[n_2],\, e_{k_1'}[n_1]e_{k_2'}[n_2]\rangle &= \sum_{n_1,n_2} e_{k_1}[n_1]e_{k_2}[n_2]e^*_{k_1'}[n_1]e^*_{k_2'}[n_2] \\ &= \left(\sum_{n_1} e_{k_1}[n_1]e^*_{k_1'}[n_1]\right)\left(\sum_{n_2} e_{k_2}[n_2]e^*_{k_2'}[n_2]\right) \\ &= \langle e_{k_1},\, e_{k_1'}\rangle\langle e_{k_2},\, e_{k_2'}\rangle = \delta[k_1 - k_1']\delta[k_2 - k_2'].\end{aligned}$$

Exercise 3.24. We define the sub-images

$$\forall\, 0 \leqslant k_1, k_2 < L, \quad \forall\, 0 \leqslant n_1, n_2 < M, \quad f_k[n] = f[n + kM].$$

One has

$$f[n] = \sum_k f_k[n - kM],$$

so that

$$f \star h[n] = \sum_k f_k[n - kM] \star h[n] = \sum_k (\tilde{f}_k \circledast h)[n - kM]$$

where we have denoted by $\tilde{f}_k$ the image of size $(2M-1) \times (2M-1)$ obtained by zero-padding from f_k. This allows one to compute $f \star h$ using L^2 circular FFT of size $(2M-1)^2$, followed by $4N$ additions to reconstruct the full image. The overall complexity of this overlap add method is thus approximately $8KN\log(M)$.

The complexity of a direct evaluation of the convolution is $K'NM^2$, where $K' \sim 2$ (1 addition and 1 multiplication).

For $K = 6$ and $K' = 2$, using the overlap-add algorithm is better in the range of M such that

$$48N\log_2(M) \leqslant 2NM^2 \quad \Leftrightarrow \quad M^2/\log_2(M) \geqslant 24$$

which we found numerically to be $M \geqslant 9$.

Exercise 3.25. The computation of $\hat{f}$ is performed by applying a 1D FFT to each direction of the 3D array. This requires

$$KN(\log(N_1) + \log(N_2) + \log(N_3)) = KN\log(N)$$

operations.

3 Chapter 4

Exercise 4.1. a) One has

$$Sf(u,\xi) = \int e^{i\phi(t)}g(t-u)e^{-i\xi t}\mathrm{d}t = \hat{h}(\xi) \quad \text{where} \quad h(t) = e^{i\phi(t)}g(t-u).$$

Using Parseval conservation of energy and the fact that $\|g\| = 1$,

$$\int |Sf(u,\xi)|^2 \mathrm{d}\xi = 2\pi \int |h(t)|^2 \mathrm{d}t = 2\pi \int |g(t)|^2 = 2\pi.$$

b) One has, using the fact that $\hat{h}'(\xi) = i\xi \hat{h}(\xi)$ and Parseval conservation of inner product,

$$\int \xi |Sf(u,\xi)|^2 \mathrm{d}\xi = \int \xi \hat{h}(\xi) \hat{h}^*(\xi) \mathrm{d}\xi = 2i\pi \int h'(t) h^*(t) \mathrm{d}t,$$

and thus, expanding $h'(t)h^*(t)$,

$$\int \xi |Sf(u,\xi)|^2 \mathrm{d}\xi = 2\pi \int \phi'(t) |g(t-u)|^2 \mathrm{d}t + 2\pi \int g'(t-u) g^*(t-u) \mathrm{d}t,$$

and the second integral vanishes because g' is an odd function.

If one interpret $P_u(\xi) = |Sf(u,\xi)|^2/(2\pi)$ as a probability density then this result proves that the average frequency of this density for a fixed t is equal to the averaged instantaneous frequency $\phi'(u)$ over a neighborhood of t defined by the density $|g(t-u)|^2$.

Exercise 4.2. The formula (2.32) for the Fourier transform of the Gaussian implies

$$\begin{aligned} Ag(\tau,\gamma) &= \frac{1}{\sqrt{\pi\sigma^2}} \int \exp\left(\frac{1}{2\sigma^2} \left[(v+\tau/2)^2 + (v-\tau/2)^2 \right] \right) e^{-i\gamma v} \mathrm{d}v \\ &= \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\tau^2}{4\sigma^2}} \int e^{-v^2} e^{-i\gamma v} \mathrm{d}v = \exp\left(-\frac{\tau^2}{4\sigma^2} - \frac{\gamma^2\sigma^2}{4} \right) \end{aligned}$$

Exercise 4.3. The reconstruction (4.66) is exact if and only if for all signal f,

$$f = \frac{\log(a)}{C_\psi} \sum_{j=1}^{J} \frac{1}{a^j} f \circledast \psi_j^* \circledast \psi_j + \frac{1}{a^J C_\psi} f \circledast \phi_J^* \circledast \phi_J.$$

Taking the finite discrete Fourier transform of this relation leads to

$$\forall\, k, \quad \hat{f}[k] = \hat{f}[k] \cdot \left(\frac{\log(a)}{C_\psi} \sum_{j=1}^{J} \frac{1}{a^j} |\psi_j[k]|^2 + \frac{1}{a^J C_\psi} |\phi_J[k]|^2, \right)$$

and hence the result.

Exercise 4.4. One can write the discrete Fourier transform in (4.27)

$$Sf[m,l] = e^{-i2\pi lm/N} \, f \star g_l[m] \quad \text{with} \quad g_l[n] = g[-n]\, e^{-i2\pi ln/N}.$$

It is thus computed with N convolutions. Since g_l has a support of size L, the fast overlapp-add convolution algorithm of Section 3.3.4 computes each convolution with $O(N \log L)$ operations. The windowed Fourier transform is thus computed with $O(N^2 \log L)$ operations.

Exercise 4.5. a) Applying (4.19) to $f = g_{u_0,\xi_0}$ proves

$$g_{u_0,\xi_0} = \frac{1}{2\pi} \iint \langle g_{u_0,\xi_0},\, g_{u,\xi} \rangle g_{u,\xi} \mathrm{d}\xi \mathrm{d}u.$$

Taking the inner product with $g_{v,\nu}$ proves

$$K(u_0, v, \xi_0, \nu) = \frac{1}{2\pi} \iint K(u_0, u, \xi_0, \xi) K(u, v, \xi, \nu) \mathrm{d}u \mathrm{d}\xi,$$

which implies $PP\Phi = P\Phi$ so that P is a projector on $\mathrm{Im}(P)$, which is exactly, by Theorem 4.2 the functions $\Phi(u, \xi)$ that are windowed Fourier transform of some $f \in \mathrm{L}^2$.

The projector P is symmetric because

$$K(u_0, u, \xi_0, \xi) = K(u, u_0, \xi, \xi_0)^*.$$

A projector P is orthogonal (which means that $\mathrm{Ker}(P) \perp \mathrm{Im}(P)$) if and only it is a symmetric operator $P = P^*$ where P^* is the dual (conjugated transposed) operator of P for the inner product

$$\langle \Phi_1, \, \Phi_2 \rangle = \frac{1}{2\pi} \iint \Phi_1(u, \xi) \Phi_2(u, \xi)^* \mathrm{d}u \mathrm{d}\xi.$$

Indeed for all $\Phi_1 \in \mathrm{Ker}(P)$ and $\Phi_2 = P\Phi_2 \in \mathrm{Im}(P)$,

$$\langle \Phi_1, \, \Phi_2 \rangle = \langle \Phi_1, \, P\Phi_2 \rangle = \langle P\Phi_1, \, \Phi_2 \rangle = 0.$$

b) Since an orthogonal projector P is 1-Lipschitz, computing $P\tilde{S}f$ reduces the quantization error because

$$\|Sf - P\tilde{S}f\| = \|P(Sf - \tilde{S}f)\| \leqslant \|Sf - S\tilde{f}\|.$$

Exercise 4.6. One has

$$b(t) = \frac{1}{C_\psi} \int_0^{s_0} f \star \psi_s^* \star \psi_s(t) \frac{\mathrm{d}s}{s^2} + \frac{1}{C_\psi s_0} f \star \phi_{s_0}^* \star \phi_{s_0}.$$

Taking the Fourier transform leads to

$$\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C_\psi} \left[\int_0^{s_0} |\hat{\psi}_s(\omega)|^2 \frac{\mathrm{d}s}{s^2} + \frac{1}{s_0} |\hat{\phi}_{s_0}|^2 \right].$$

Using the fact that $|\hat{\psi}_s(\omega)|^2 = s|\hat{\psi}(s\omega)|^2$ and

$$|\hat{\phi}_{s_0}|^2 = \int_1^{+\infty} |\hat{\psi}(s_0 s\omega)|^2 \frac{\mathrm{d}s}{s} = \int_{s_0}^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{\mathrm{d}s}{s}$$

leads to the result.

Exercise 4.7. Using Plancherel formula and then Fubini,

$$\begin{aligned} \|\phi\|^2 &= \frac{1}{2\pi} \int |\hat{\phi}(\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi} \int \int_1^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{\mathrm{d}s}{s} \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_1^{+\infty} \left(\int |\hat{\psi}(s\omega)|^2 \mathrm{d}\omega \right) \frac{\mathrm{d}s}{s}. \end{aligned}$$

The change of variable $s\omega \to s$ in the inner integral, and the fact that $\|\psi\| = 1$ leads to

$$\|\phi\|^2 = \int_1^{+\infty} \left(\frac{1}{2\pi} \int |\hat{\psi}(\omega)|^2 \mathrm{d}\omega \right) \frac{\mathrm{d}s}{s^2} = \int_1^{+\infty} \frac{\mathrm{d}s}{s^2} = 1.$$

Exercise 4.8. Let

$$b(t) = \frac{1}{C}\int_0^{+\infty} f\star\psi_s(t)\frac{\mathrm{d}s}{s^{3/2}}.$$

Its Fourier transform reads, after a change of variable $\xi = s\omega$,

$$\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C}\int_0^{+\infty}\sqrt{s}\hat{\psi}(s\omega)\frac{\mathrm{d}s}{s^{3/2}} = \frac{\hat{f}(\omega)}{C}\int_0^{+\infty}\hat{\psi}(\xi)\frac{\mathrm{d}s}{s} = \hat{f}(\omega).$$

Exercise 4.9. a) If f is $\mathbf{C}^p$, then $\hat{f}^{(p)}(\omega) = (i\omega)^p\hat{f}(\omega)$ and thus the inverse Fourier transform formula gives the result.
b) On the set $\Omega_\rho = \{z \setminus \mathrm{Im}(z) > \rho\}$, one has

$$|(i\omega)^p\hat{f}(\omega)e^{iz\omega}| \leqslant |\omega|^p|\hat{f}(\omega)|e^{-\rho\omega},$$

which is integrable. Using classical result of holomorphy under the sign $\int$, one sees that $f^{(p)}(z)$ is holomorphic on Ω_ρ for all $\rho > 0$.
c) One has

$$f^{(p)}(x+iy) = \frac{1}{2\pi}\int_0^{+\infty}(i\omega)^p\hat{f}(\omega)e^{ix\omega}e^{-y\omega}\mathrm{d}\omega.$$

A wavelet transform of f is written over the Fourier domain

$$Wf(u,s) = \frac{1}{2\pi}\int\hat{f}(\omega)\sqrt{s}\hat{\psi}(s\omega)e^{i\omega u}\mathrm{d}\omega$$

so one sees that $f^{(p)}$ is indeed a Fourier transform if one sets

$$\hat{\psi}(\omega) = 1_{[0,+\infty[}(\omega)(i\omega)^p e^{-\omega}.$$

Using the inverse Fourier transform, and the fact that $\hat{h}^{(p)}(\omega) = (i\omega)^p\hat{h}(\omega)$, one has

$$\psi(t) = h^{(p)}(t) \quad \text{where} \quad h(t) = \frac{1}{2\pi}\int_0^{+\infty}e^{-\omega}e^{i\omega t} = \frac{1}{2\pi}\frac{1}{1-it},$$

so

$$\psi(t) = \frac{ip!}{2\pi(t+i)^p}.$$

Exercise 4.10. For $f(t) = \cos(\theta(t))$ with $\theta(t) = a\cos(bt)$, the width s of the window is small enough if

$$s^2|\theta''(t)| = s^2ab^2|\cos(bt)| \ll 1$$

and thus $s^2ab^2 \ll 1$.

For $\theta(t) = \cos(\theta_1(t)) + \cos(\theta_2(t))$ with $\theta_1(t) = a\cos(bt)$ and $\theta_2(t) = a\cos(bt) + ct$, there is enough frequency resolution if

$$|\theta_1'(t) - \theta_2'(t)| = |c| \geqslant \frac{\Delta_\omega}{s}$$

and enough spacial resolution if

$$s^2|\theta_1'(t)| = s^2|\theta_2''(t)| = s^2ab^2 \ll 1.$$

This shows that one needs

$$\sqrt{\frac{c}{ab^2}} \ll \Delta_\omega \simeq 1.$$

Exercise 4.15. One has,

$$\int Pf(u,\xi)\mathrm{d}u = \frac{1}{\|f\|^2}\int |f(u)|^2\mathrm{d}u|\hat{f}(\omega)|^2 = |\hat{f}(\omega)|^2$$

Using Plancherel conservation of energy formula, one has

$$\frac{1}{2\pi}\int Pf(u,\xi)\mathrm{d}\xi = |f(u)|^2\frac{1}{\|f\|^2}\int |\hat{f}(\omega)|^2\mathrm{d}\omega = |f(u)|^2.$$

Since Pf is not quadratic in f (it is of degree 4), one cannot apply Theorem 4.11.

Exercise 4.16. Example 4.20 shows that $P_\theta f$ is a spetrogram with a Gaussian window g_μ if

$$\theta(u,\xi) = g_\sigma(u)g_\beta(\xi) = P_V g_\mu(u,\xi) = g_{\mu/\sqrt{2}}(u)g_{\sqrt{2}\mu}(\xi),$$

where we have used (4.125) for the last equality. If $\sigma\beta = 1/2$, then one can take $\mu = \sqrt{2}\sigma$. Otherwise, if $\sigma\beta > 1/2$, one decomposes $\theta = \theta_0 \star \theta_1$ where

$$\theta_0(u,\xi) = g_{\mu/\sqrt{2}}(u)g_{\sqrt{2}\mu}(\xi) \quad \text{and} \quad \theta_1(u,\xi) = g_{\sigma-\mu/\sqrt{2}}(u)g_{\beta-\sqrt{2}\mu}(\xi)$$

where μ is chosen so that $\sigma - \mu/\sqrt{2} > 0$ and $\beta - \sqrt{2}\mu > 0$. This shows that P_θ is a smoothing with θ_1 of P_{θ_0} which is a spectrogram and hence positive.

Exercise 4.17. Since $\{g_n(t)\}_{n\in\mathbb{N}}$ is an orthonormal basis,

$$\sum_{n=0}^{+\infty} g_n^*(t)\, g_n(t') = \delta(t-t') \ .$$

Indeed, for all $f \in \mathrm{L}^2$

$$\begin{aligned}\int f(t)\sum_{n=0}^{+\infty} g_n^*(t)\, g_n(t')\, dt &= \sum_{n=0}^{+\infty}\langle f,\, g_n\rangle g_n(t') \\ &= f(t') = \int f(t)\,\delta(t-t')dt \ .\end{aligned}$$

It results that

$$\sum_{n=0}^{+\infty} g_n^*(u-\tau/2)\, g_n(u+\tau/2) = \delta(\tau) \ .$$

Since $PV g_n(u,\xi)$ is the Fourier transform of $g_n^*(u-\tau/2)\, g_n(u+\tau/2)$ with respect to τ, it results that

$$\sum_{n=0}^{+\infty} P_V g_n(u,\xi) = 1 \ .$$

Exercise 4.18. One has

$$A(t) = \int (\xi-\phi'(t))^2 P_V f(t\xi)\mathrm{d}\xi = \int h(\tau)\left(\int (\xi-\phi'(t))^2 e^{-i\xi\tau}\mathrm{d}\xi\right)\mathrm{d}\tau$$

where

$$h(\tau) = a(t+\tau/2)a(t-\tau/2)e^{i[\phi(t+\tau/2)-\phi(t-\tau/2)]}.$$

In the sense of distribution, one has the following equality

$$\int (\xi - \phi'(t))^2 e^{-i\xi\tau} \mathrm{d}\xi = 2\pi[-\delta''(\tau) + 2i\phi'(t)\delta'(\tau) + \phi'(t)^2\delta(\tau)]$$

where $\delta, \delta', \delta''$ are the Dirac distribution and its first and second derivatives. This shows that

$$A(t) = 2\pi[-h''(0) + 2i\phi'(t)h'(0) + \phi'(t)^2 h(0)].$$

After computing the derivatives of h and simplification, one finds

$$A(t) = -\pi(a(t)a''(t) - a'(t)^2),$$

which is the result.

4 Chapter 5

Exercise 5.1. One has a union of K orthogonal bases

$$\mathcal{D} = \{\phi_p\}_{0\leqslant p<KN} = \bigcup_{i=0}^{K-1} \{\phi_{Kp+i}\}_{0\leqslant p<N},$$

so that $\mathcal{D}$ is tight frame of frame bound KN since

$$\sum_{p=0}^{KN-1} |\langle f,\, \phi_p\rangle|^2 = \sum_{i=0}^{K-1} N \left(\sum_{p=0}^{N-1} |\langle f,\, \phi_{pK+i}/\sqrt{N}\rangle|^2 \right) = \sum_{i=0}^{K-1} N\|f\|^2 = KN\|f\|^2.$$

Exercise 5.2. A change of variable $t \to Kt$ gives

$$\langle f,\, \phi_p\rangle = \int_0^1 f(t)e^{2i\pi pt/K}\mathrm{d}t = K\int_0^{1/K} f(Kt)e^{2i\pi pt} = \int_0^1 f_K(t)e_p(t)\mathrm{d}t,$$

where $\{e_p(t) = e^{-2i\pi t}\}_p$ is an orthonormal basis of $\mathrm{L}^2[0,1]$. Let us define $f_K(t) = f(Kt)$ for $t \in [0, 1/K]$ and $f_K(t) = 0$ otherwise. For $K \geqslant 1$, we get

$$\sum_p |\langle f,\, \phi_p\rangle|^2 = \sum_p |\langle f_K,\, e_p\rangle|^2 = \|f_K\|^2 = K\|f\|^2$$

So $\{\phi_p\}_p$ is a tight frame of $\mathrm{L}^2[0,1]$ of frame bound K.

Exercise 5.3. The operator $\Phi\Phi^*$ is bounded on $H = \mathrm{Im}(\Phi)$ which is of finite dimension, so $B < +\infty$. If $A = 0$, then it exists $f \neq 0$ in H such that $\sum_n |\langle f,\, \phi_n\rangle|^2 = 0$ so that $\langle f,\, \phi_n\rangle = 0$ for all n. Since $\{\Phi_n\}_n$ is generator of H, one has $f = 0$ which is a contradiction.

Exercise 5.4. One has

$$\mathrm{tr}(U_1U_2) = \sum_{i,j} U_1[i,j]U_2[j,i] = \mathrm{tr}(U_2U_1).$$

Exercise 5.5. One has $\Phi f[p] = f \circledast h$ where $h = \delta[\cdot] - \delta[\cdot - 1]$. As the discrete Fourier basis diagonalizes Φ, the frame bounds are

$$A = \min_{\omega\neq 0} |\hat{h}[\omega]|^2 = \min_{\omega\neq 0} 4\sin^2\left(\frac{\pi}{N}\,\omega\right) = 4\sin^2(\pi/N)$$

and $B = 4$ (if N is even). One has $A/B \to 0$ when $N \to +\infty$, so the frame is unstable when N is large.

Exercise 5.6. If one does not constrain $\|\phi_p\|$, one can choose $\{\delta_0/N, N\delta_1, \delta_2 \dots, \delta_{N-1}\}$. If one constrains $\|\phi_p\| = 1$, one can choose $\phi_p[n] = \phi[n + p \mod N]$ so that $\Phi f = f * \phi$ and thus one should impose

$$\sum_\omega |\hat{\phi}[\omega]|^2 = N \quad \text{and} \quad |\hat{\phi}[\omega]| > 0.$$

This can be achieved by setting $\hat{\phi}[\omega] = 1/N$ for $\omega \neq 0$ and $\hat{\phi}[0] = \sqrt{N - (N-1)/N^2}$. This implies

$$A = \min_\omega |\hat{\phi}[\omega]|^2 = 1/N^2 \to 0$$

and

$$B = \max_\omega |\hat{\phi}[\omega]|^2 = N - (N-1)/N^2 \to +\infty.$$

Exercise 5.7. If $x \in \mathrm{Null}(U^*)$ and $y = Uz \in \mathrm{Im}(U)$, then

$$\langle x, y \rangle = \langle x, Uz \rangle = \langle U^* x, z \rangle = 0$$

so that $\mathrm{Null}(U^*) = \mathrm{Im}(U)^\perp$.

Exercise 5.8. One has

$$\Phi_m f[p] = \langle f, \phi_{m+p} \rangle = f \circledast \phi_m[p] \quad \Longrightarrow \quad \Phi^* \Phi = \sum_m \Phi_m^* \Phi_m$$

where Φ_m is a convolution operator, whose eigenvalues are $\hat{\phi}_m[\omega]$, and eigenvectors are the discrete Fourier vectors. The eigenvalues of $\Phi^*\Phi$ are thus $\sum_m |\hat{\phi}_m[\omega]|^2$.

Exercise 5.9. Let

$$\phi_{k,p}(t) = g(t - 2p\pi/\omega_0) e^{ik\omega_0 t} \quad \text{so} \quad \hat{\phi}_{k,p}(\omega) = \hat{g}(\omega - k\omega_0) e^{i\omega/\omega_0 2p\pi}.$$

Since $\{e^{i\omega/\omega_0 2p\pi}\}_p$ is an orthonormal basis of each interval $[-\omega_0/2, \omega_0/2] + k\omega_0$, $\{1/\sqrt{2\pi}\hat{\phi}_{k,p}\}_{k,p}$ is an orthogonal basis of $\mathrm{L}^2(\mathbb{R})$. Since $f \mapsto 1/\sqrt{2\pi}\hat{f}$ is an isometry, this prove that $\{\phi_{k,p}\}_{k,p}$ is also an orthogonal basis.

Exercise 5.10. a) Since $\{1/\sqrt{K} e^{\frac{2i\pi}{K} nk}\}_{0 \leqslant k < K}$ is an orthogonal basis of the signal supported in $I = [-K/2, K/2 - 1] + mM$, and $g[n - mM] f[n]$ is supported in I, one has

$$\sum_n |g[n - mM]|^2 |f[n]|^2 = \sum_{k=0}^{K-1} |\langle f, \frac{1}{\sqrt{K}} g_{mnk} \rangle|^2.$$

b) Summing over m leads to

$$K \sum_n |f[n]|^2 \sum_{m=0}^{N/M-1} |g[n - mM]|^2 = \sum_{k,m} |\langle f, g_{m,k} \rangle|^2,$$

and hence the result of Theorem 5.18.

Exercise 5.11. For $m = 2$, the filters are written, with the convention $z = e^{-i\omega}$

$$\hat{h}(\omega)/\sqrt{2} = \cos^3(\omega/2) e^{-i\omega/2} = P(z) = (z^{-1} + 3 + 3z + z^2)/8$$

$$\hat{g}(\omega)/\sqrt{2} = -i\sin(\omega/2)e^{-i\omega/2} = Q(z) = z/2 - 1/2.$$

Applying Bezout algorithm to the polynomials $(1+3z+3z^2+z^3)/8$ and $(z^2-z)/2$ leads to

$$(8z - 7z^2) \times P(z) + (\frac{17}{4}z + 5z^2 + \frac{7}{4}z^3)Q(z) = 1$$

and the reconstruction filters are given by

$$\hat{\tilde{h}}(\omega)/\sqrt{2} = 8z^{-1} - 7z^{-2} \quad \text{and} \quad \hat{\tilde{g}}(\omega) = \frac{17}{4}z^{-1} + 5z^{-2} + \frac{7}{4}z^{-3}.$$

Exercise 5.12. For $m = 3$, one has

$$\begin{aligned}\hat{h}(\omega)/\sqrt{2} &= \cos^4(\omega/2)^4 = P(z) = (z/2 + z^{-1}/2)^4 \\ &= (z^2 + 4z + 6 + 3z^{-1} + z^{-2})/2^4\end{aligned}$$

and one can choose

$$\hat{g}(\omega)/\sqrt{2} = \sin^2(\omega/2) = Q(z) = (z - 2 + z^{-1})/4.$$

If $\tilde{h} = h$, then

$$\hat{\tilde{g}}(\omega) = \frac{2 - |\hat{h}(\omega)|^2}{\hat{g}^*(\omega)} = 2\frac{1 - \cos^4(\omega/2)}{\sqrt{2}\sin^2(\omega/2)} = \sqrt{2}[1 + \cos^2(\omega/2)] = \sqrt{2}\tilde{Q}(z)$$

where $\tilde{Q}(z) = 1 + (z + 2 + z^{-1})/4$, so $\tilde{g}/\sqrt{2} = [1/4, 3/2, 1/4]$.

Exercise 5.13. Using the same proof as Theorem 5.18 and exercise 5.10, with $M = 1$, $K = N$, one sees that

$$\left\{ g_{m,\ell}[n] = g[n-m]\exp\left(\frac{2i\pi}{N}\ell n\right)\right\}_{m,\ell}$$

is a tight frame of frame bound

$$N\sum_n |g[n-m]|^2 = N\|g\|^2 = N.$$

Theorem 5.5 implies that the dual frame is $\{g_{m,\ell}/N\}_{m,\ell}$.

The reproducing kernel reads

$$K(m, m', \ell, \ell') = \langle g_{m,\ell},\, g_{m',\ell'}\rangle = \sum_n g[n-m]g[n-m']\exp\left(\frac{2i\pi}{N}n(\ell - \ell')\right).$$

Exercise 5.14. As $\beta\eta \to 2\pi$, the frame becomes less orthogonal ($A/B \to +\infty$), so the dual vectors become more and more different from the primal ones, and hence the windows g and $\tilde{g}$ differ more and more.

Exercise 5.15. a) One has

$$\forall \pi \in [-\pi, \pi], \quad \hat{x}(\omega) = \frac{1}{s}\hat{f}(\omega/s)$$

for $s = T/K$, so $\mathrm{supp}(\hat{x}) \subset [-\pi/K, \pi/K]$.

b) Since $x - \tilde{x} = W$

$$J(h) = E(\|\tilde{x} \star h - x\|^2) = E(\|x \star (h - \delta) + W \star h\|^2).$$

Since W is a white noise of variance σ^2 independant of x

$$J(h) = E(\|x \star (h - \delta)\|^2) + E(\|W \star h\|^2).$$

Since W is a white noise of variance σ^2 its power spectrum is σ^2. Let $\hat{R}_x(\omega)$ be the power spectrum of the stationary random vector $x[n]$,

$$J(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\hat{R}_x(\omega)|1 - \hat{h}(\omega)|^2 + \sigma^2 |\hat{h}(\omega)|^2)\, d\omega \ .$$

To minimize $J(h)$ for each ω we minimize the value under the integral, which is obtained with

$$\hat{h}(\omega) = \frac{\hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega)} \quad \text{and} \quad J(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma^2 \hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega)} \mathrm{d}\omega \leqslant \frac{\sigma^2}{K}.$$

c) In this case $\tilde{x} = x \star h_p + W$ so

$$\begin{aligned} J(h) &= E(\|x \star (h \star h_p - \delta)\|^2) + E(\|W \star h\|^2) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\hat{R}_x(\omega)|1 - \hat{h}(\omega)\, \hat{h}_p(\omega)|^2 + \sigma^2 |\hat{h}(\omega)|^2)\, d\omega \ . \end{aligned}$$

The minimum is obtained with

$$\hat{h}(\omega) = \frac{\hat{R}_x(\omega)\, \hat{h}_p(\omega)}{\sigma^2 + \hat{R}_x(\omega)\, |\hat{h}_p(\omega)|^2} \quad \text{and} \quad J(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma^2 \hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega) |h_p(\omega)|^2} \mathrm{d}\omega.$$

Since $\hat{h}_p(\omega) = (1 - e^{-i\omega})^{-p}$,

$$J(h) \leqslant \frac{\sigma^2}{2\pi} \int_{-\pi/K}^{\pi/K} |2 \sin(\omega/2)|^{2p} \mathrm{d}\omega \leqslant \frac{\sigma^2}{K} \, \frac{\pi^{2p}}{K^{2p}\,(2p+1)}.$$

For a fixed $K \geqslant \pi$, increasing p decreases the error.

Exercise 5.16. a) Each $\{\phi_s(t - ns - ks/K)\}_n$ is an orthogonal basis of U_s, the set of signals with Fourier support included in $[-\pi/s, \pi/s]$. This proves that $\{\phi_{s-n/Ks}\}_n$ is a tight frame with frame bound K.
b) One has

$$Pa[n] = \sum_{n'} a[n'] H[n, n'] \quad \text{where} \quad H[n, n'] = \langle \phi_s(t - n/Ks),\, \frac{1}{K} \phi_s(t - n'/Ks) \rangle,$$

and, using Plancherel formula,

$$\begin{aligned} H[n, n'] &= \frac{1}{2K\pi} \langle s^{1/2} 1_{[-\pi/s, \pi/s]}(\omega) e^{-in/Ks\omega},\, s^{1/2} 1_{[-\pi/s, \pi/s]}(\omega) e^{-in'/Ks\omega} \rangle \\ &= \frac{s}{2K\pi} \int_{-\pi/s}^{\pi/s} e^{i(n-n')/Ks\omega} \mathrm{d}\omega = h[n - n'] \quad \text{where} \quad h[n] = \frac{\sin(\pi n/K)}{\pi n}. \end{aligned}$$

c) Since $\hat{h} = 1_{[-\pi/K,\pi/K]}$, one has

$$\mathrm{Im}(P) = \{a \setminus h \star a = a\} = \{a \setminus \mathrm{supp}(\hat{a}) \subset [-\pi/K, \pi/K]\}.$$

d) Theorem 3.5 proves that $s^{-1/2} f \star \phi_s(ns) = f(ns)$.
e) Let $a[n] = f \star \phi_s(ns_0)$. For $\omega \in [-\pi, \pi]$, one has

$$\hat{a}(\omega) = \frac{1}{s_0}\mathcal{F}(f \star \phi_s)(\omega/s_0) \quad \Longrightarrow \quad \mathrm{supp}(\hat{a}) \subset [-\pi/K, \pi/K]$$

where $\mathcal{F}$ is the Fourier transform. This shows that $a \in \mathrm{Im}(\Phi)$ and hence $Pa = a$.
One has

$$PY[Kn] = a[Kn] + P(W)[Kn] \quad \Longrightarrow \quad |PY[Kn] - s^{-1/2} f(ns)|^2 = |W \star h[Kn]|^2.$$

One then has

$$E(|W \star h[Kn]|^2) = \sigma^2 \sum_k |h[k]|^2 = \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} |\hat{h}(\omega)|^2 = \frac{\sigma^2}{K} \to 0 \quad \text{when} \quad K \to +\infty.$$

Exercise 5.17. a) Theorem 5.11 applied to the generator $\frac{1}{2^{j/2}\psi(u/2^j)} = \phi_n(u)$ proves the result.
b) ψ has p vanishing moments if and only if $\hat{\psi}^{(k)}(0) = 0$ for all $k < p$. Thus, since

$$\sum_j |\hat{\psi}(2^j\omega)|^2 > A > 0,$$

one has that $\hat{\tilde{\psi}}^{(k)}(0) = 0$ and thus $\tilde{\psi}$ has also p vanishing moments.
c) One has for $g = \{g_j(u)\}_j$

$$(P_V g)_j(u) = \sum_{j'} \int g_{j'}(u')\langle \psi_{2^j,u}, \tilde{\psi}_{2^{j'},u'}\rangle \mathrm{d}u' = \sum_{j'} \int g_{j'}(u') \cdot (\psi_j \star \tilde{\psi}_{j'})(u - u')\mathrm{d}u.$$

This shows that

$$(P_V g)_j = \psi_j \star \sum_{j'} g_{j'} \star \tilde{\psi}_{j'}.$$

Exercise 5.18. We prove the right hand side of the inequality

$$\begin{aligned}\sum_{j=\alpha}^{\beta} |\hat{\psi}(2^j\omega)|^2 &= \frac{1}{2}\sum_{j=\alpha}^{\beta} |\hat{g}(2^{j-1}\omega)|^2 |\hat{\phi}(2^{j-1}\omega)|^2 \\ &\leqslant B \sum_{j=\alpha}^{\beta} (1 - |\hat{h}(2^{j-1}\omega)|^2)|\hat{\phi}(2^{j-1}\omega)|^2 \\ &\leqslant B \sum_{j=\alpha}^{\beta} (|\hat{\phi}(2^{j-1}\omega)|^2 - |\hat{\phi}(2^j\omega)|^2) = B(|\hat{\phi}(2^{\alpha-1}\omega)|^2 - |\hat{\phi}(2^{\beta}\omega)|^2).\end{aligned}$$

Since $\hat{\phi}(0) = 1$ and $\phi(\omega)$ tends to zero as ω increases, letting α and β go to $-\infty$ and $+\infty$ proves that $\sum_j |\hat{\psi}(2^j\omega)|^2 \leqslant B$. A similar proof applies to the left hand side.

Exercise 5.19. a) One has

$$Zg_{n,k}(u,\xi) = \sum_{\ell} e^{2i\pi\xi\ell} 1_{[0,1]}(u-\ell-n) e^{2i\pi k(u-\ell)}.$$

Since $1_{[0,1]}(u-\ell-n)$ is zero unless $\ell = -n$, one obtains

$$Zg_{n,k}(u,\xi) = e^{-2i\pi\xi n} e^{2i\pi uk}.$$

Since $\{e^{-2i\pi\xi n} e^{2i\pi uk}\}_{n,k}$ is an orthogonal basis of $\mathrm{L}^2[0,1]^2$, the Zak transform transforms an orthogonal basis into an orthogonal basis, and is thus an unitary transforms.

b) One has

$$\int_0^1 Zf(u,\xi)\mathrm{d}\xi = \sum_{\ell} \int_0^1 e^{2i\pi\xi\ell}\mathrm{d}\xi f(u-\ell) = f(u).$$

Note that this formula is valid to recover $f(u)$ for $u \in [0,1]$, but it can be extended to arbitrary $u \in \mathbb{R}$ using the fact that

$$Zf(u+1,\xi) = e^{2i\pi\xi} Z(u,\xi).$$

c) Let $g_{n,k}(t) = e^{2i\pi k} g(t-n)$. Similarly to question a), one proves that

$$Zg_{n,k}(u,\xi) = e^{-2i\pi\xi n} e^{2i\pi uk} (Zg)(u,\xi).$$

One has, using the fact that the Zak transform is unitary, and then Plancherel formula for Fourier series of functions defined on $[0,1]^2$,

$$\sum_{n,k} |\langle f,\, g_{n,k}\rangle|^2 = \sum_{n,k} |\langle Zf,\, Zg_{n,k}\rangle|^2 = \sum_{n,k} \left| \iint e^{-2i\pi\xi n} e^{2i\pi uk} (Zf)(u,\xi)(Zg)^*(u,\xi)\mathrm{d}u\mathrm{d}\xi \right|^2$$
$$= \iint |(Zf)(u,\xi)|^2\, |(Zg)(u,\xi)|^2 \mathrm{d}u\mathrm{d}\xi.$$

This shows that if

$$\forall\, (u,\xi), \quad A \leqslant |(Zg)(u,\xi)|^2 \leqslant B,$$

then, using the fact that Z is an isometry,

$$A\|f\|^2 = A\|Zf\|^2 \leqslant \sum_{n,k} |\langle f,\, g_{n,k}\rangle|^2 \leqslant B\|Zf\|^2 = B\|f\|^2,$$

which proves that $\{g_{n,k}\}_{n,k}$ is a frame with bounds A and B.

d) Denoting $\Phi f[n,k] = \langle f,\, g_{n,k}\rangle$ the frame operator, we have proved that

$$\|\Phi f\|^2 = \iint |(Zf)(u,\xi)|^2\, |(Zg)(u,\xi)|^2 \mathrm{d}u\mathrm{d}\xi,$$

so that

$$Z(\Phi^*\Phi f)(u,\xi) = |(Zg)(u,\xi)|^2 (Zf)(u,\xi)$$

and hence

$$Z((\Phi^*\Phi)^{-1} f)(u,\xi) = \frac{(Zf)(u,\xi)}{|(Zg)(u,\xi)|^2}.$$

The dual frame $\{\tilde{g}_{n,k}\}_{n,k}$ is defined as

$$\tilde{g}_{n,k} = (\Phi^*\Phi)^{-1} g_{n,k}$$

and hence

$$Z\tilde{g}_{n,k}(u,\xi) = \frac{(Zg_{n,k})(u,\xi)}{|(Zg)(u,\xi)|^2} = \frac{(Zg)(u,\xi)e^{-2i\pi\xi n}e^{2i\pi uk}}{|(Zg)(u,\xi)|^2}$$
$$= \frac{e^{-2i\pi\xi n}e^{2i\pi uk}}{(Zg)^*(u,\xi)} = (Zh_{n,k})(u,\xi)$$

where we have defined

$$h_{n,k}(t) = e^{2i\pi k}h(t-n) \quad \text{where} \quad (Zh)(u,\xi) = (Zg)^*(u,\xi)^{-1}.$$

Exercise 5.21. We use the change of variables

$$x = \rho\cos(\alpha) - u\sin(\alpha) \quad \text{and} \quad y = \rho\sin(\alpha) + u\cos(\alpha).$$

For $k+\ell < p$, one can expand the monomial $P(x,y) = x^k y^\ell$ as a polynomial in u of degree less than p

$$P(x,y) = \sum_{t<p} A(\rho)u^t.$$

This shows that

$$\langle P,\,\psi\rangle = \sum_{t<p}\int A(\rho)\left(\int u^t\psi(\rho\cos(\alpha)-u\sin(\alpha),\rho\sin(\alpha)+u\cos(\alpha))\mathrm{d}u\right)\mathrm{d}\rho = 0.$$

Exercise 5.22. a) One has

$$\sum_k n_k n_k^* = \begin{pmatrix} \mathrm{A} & \mathrm{C} \\ \mathrm{C} & \mathrm{B} \end{pmatrix},$$

where, denoting $\alpha_k = 2k\pi/K$,

$$A = \sum_k \cos^2(\alpha_k), \quad B = \sum_k \sin^2(\alpha_k) \quad \text{and} \quad C = \sum_k \cos(\alpha_k)\sin(\alpha_k).$$

One has $C = 0$ and

$$A = B = \frac{K}{2} + \sum_k \cos(2\alpha_k) = \frac{K}{2},$$

so $\sum_k n_k n_k^* = \frac{K}{2}\mathrm{Id}_2$, which proves that $\{n_k\}_k$ is a tight frame with frame bound $K/2$.
b) One has

$$\psi^k = \langle[\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y}],\,n_k\rangle \quad \Longrightarrow \quad \hat{\psi}^k(\omega) = \hat{\theta}(\omega)\langle i[\omega_x,\omega_y],\,n_k\rangle.$$

This shows that

$$\sum_{k=0}^{K}\sum_j |\hat{\psi}^k(2^j\omega)|^2 = \sum_j |\hat{\theta}(\omega)|^2 2^{2j}\sum_k |\langle[\omega_x,\omega_y],\,n_k\rangle|^2$$

and hence the result since

$$\sum_k |\langle[\omega_x,\omega_y],\,n_k\rangle|^2 = \frac{K}{2}\|\omega\|^2.$$

5 Chapter 6

Exercise 6.1. a) It is an immediate consequence of Definition 6.1 of Lipschitz and uniform Lipschitz regularity.
b) For $t \neq 0$ f is C^1 so it is 1-Lipschitz, and $|f(t)| \leqslant |t|$ so f is also 1-Lipschitz at 0.

If f is uniformly Lipschitz α then there exists $K > 0$ such that for all $(u, v) \in [-1, 1]^2$, $|f(u) - f(v)| \leqslant K\,|u - v|^\alpha$. For $t_n = (n + 1/2)^{-1}\pi^{-1}$ we have $f(t_n) = (-1)^n\, t_n$. So

$$|f(t_n) - f(t_{n-1})| = t_n + t_{n-1} = \pi^{-1}\left((n+1/2)^{-1} + (n-1/2)^{-1}\right) \sim 2\pi^{-1}\, n^{-1}\ .$$

Since $t_n - t_{n-1} = \pi^{-1}\,(n+1/2)^{-1}\,(n-1/2)^{-1} \sim \pi^{-1}n^{-2}$ it results that $|f(t_n) - f(t_{n-1})| \sim |t_n - t_{n-1}|^{1/2}$ and hence that $\alpha \geqslant 1/2$.

We now prove that f is indeed Lipschitz 1/2. If u and v have same sign and $|1/u - 1/v| > 1$ then there exists $C > 0$ with

$$|f(u) - f(v)| \leqslant |f(t_n) - f(t_{n-1})| \leqslant C\,|t_n - t_{n-1}|^{1/2} \leqslant C\,|u - v|^{1/2}\ .$$

If u and v of same sign with $|1/u - 1/v| \leqslant 1$ and $|u| > |v|$. It result that $|u - v| \leqslant u^2$, and

$$|f(u) - f(v)| \leqslant |u - v| + |v|(|\sin u^{-1}| - |\sin v^{-1}|) \leqslant |u - v| + |v|(v^{-1} - u^{-1}|.$$

Since $|u - v| \leqslant 1$

$$|f(u) - f(v)| \leqslant |u - v|^{1/2} + \frac{|u - v|}{|u|} \leqslant 2|u - v|^{1/2}\ .$$

If u and v have different signs, and $|f(u)| \geqslant |f(v)|$, since $|u - v| \leqslant 2$ it results that

$$|f(u) - f(v)| \leqslant 2|f(u)| \leqslant 2|u| \leqslant 2|u - v| \leqslant 2\sqrt{2}|u - v|^{1/2}.$$

It results that for any $(u, v) \in [-1, 1]^2$ there exists $C' > 0$ with

$$|f(u) - f(v)| \leqslant C'\,|u - v|^{1/2}$$

and hence that f is uniformly Lipschitz 1/2 on $[-1, 1]$.

Exercise 6.2. a) The definition of f being α Lipschitz can be written

$$f(x) = f(u) + a(u)(x - u) + K(x, u)(x - u)^\alpha \tag{2}$$

with $|K(x, u)|$ uniformly bounded in both x and u, so

$$[f(x) - f(u)]/(x - u) = a(u) + K(x, u)(x - u)^{\alpha - 1}$$

which implies that f has derivatives at u with $a(u) = f'(u)$, so (2) is re-written

$$f(x) = f(u) + f'(u)(x - u) + K(x, u)(x - u)^\alpha \tag{3}$$

Then summing (2) for $x = a + d$, $x = a$ and $x = a - d$ shows that

$$(f(a + d) - 2f(a) + f(a - d))/d = O(d^{\alpha - 1})$$

then applying (3) for $x = a$ and both $u = a - d$ and $u = a + d$ shows that

$$|f'(a - d) - f'(a + d)|/d = O(d^{\alpha - 1})$$

this shows that f' is Lipschitz $\alpha - 1$ and that this is uniform with the same bound.
b) If $f(t) = t^2 \cos(1/t)$ then $f'(t) = 2t\cos(1/t) - \sin(1/t)$. One has $|f(t) - f(0)| \leqslant t^2$ so that f is 2-Lipschitz at 0, but $\sin(1/t)$ is discontinuous at 0 of f' is not Lipschitz 1 at 0.

Exercise 6.3. One can take a hat function that is zero outsize $[-1, 1]$, that is linear over $[-1, 0]$ and $[0, 1]$, with $f(0) = 1$. It is 1-Lipschitz. We verify that $|\hat{f}(\omega)| = |\sin(\omega/2)|^2\,|\omega|^{-2}$ so (6.4) is not satisfied.

Exercise 6.4. a) One has, for $f(t) = \cos(\omega_0 t)$, and using Plancherel formula

$$\begin{aligned} Wf(u,s) &= \frac{1}{\sqrt{s}} \int \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) f(t)\,\mathrm{d}t \\ &= \sqrt{s} \int \hat{\psi}(s\omega) e^{-iu\omega} \frac{1}{2} (\delta_{\omega_0}(\omega) + \delta_{-\omega_0}(\omega))\mathrm{d}\omega \\ &= \sqrt{s}\hat{\psi}(s\omega_0)\cos(u\omega_0), \end{aligned}$$

because $\hat{\psi}(-\omega) = \hat{\psi}(\omega)$ because $\psi(t)$ is symmetric.
b) One has,

$$\frac{\partial Wf(u,s)}{\partial u} = 0 \quad \Longleftrightarrow \quad u = u_k = \frac{k\pi}{\omega_0}.$$

If ψ has p vanishing moments then $|\hat{\psi}(\omega)| = O(|\omega|^p)$ so

$$|Wf(u_k, s)| \leqslant \sqrt{s}\,|\psi(s\omega_0)| = O(s^{p+1/2}).$$

Exercise 6.5. For $f(t) = |t|^\alpha$, one has, after the change of variable $t \to t/s$,

$$Wf(u,s) = \frac{1}{\sqrt{s}} \int |t|^\alpha \psi(t/s - u/s)\mathrm{d}t = s^{\alpha+1/2} \int |t|^\alpha \psi(t - u/s)\mathrm{d}t = Wf(u/s, 1).$$

If $\psi(t)$ is an antisymmetric wavelet, $\psi(-t) = \psi(t)$, since $f(t)$ is even the wavelet transfrom is antisymmetric and hence $Wf(0,s) = 0$ for all s. In this case we thus can not derive the Lipschitz regularity at $t = 0$ from $Wf(0,s)$.

Exercise 6.6. One has $|f(t)| \leqslant |t|^\alpha$ so f is α-Lipschitz. One has for $t > 0$

$$f'(t) = \alpha t^{\alpha-1}\sin(1/t^\beta) + \beta t^{\alpha-\beta-1}\cos(1/t^\beta),$$

so f' is $\alpha - \beta - 1$-Lipschitz.

The signal can be written $f(t) = a(t)\cos\theta(t)$ with $a(t) = |t|^\alpha$ and $\theta(t) = |t|^{-\beta}$. The instantaneous frequency is $\theta'(t) = \beta\,|t|^{-\beta-1}$. For such amplitudes and instantaneous frequencies one can apply the wavelet ridge calculations. Equation (4.109) shows that the larges wavelet coefficients are along ridge curves $(u, s(u))$ defined by

$$s(u) = \eta\theta'(u)^{-1} = \eta\,\beta^{-1}\,|u|^{\beta+1}$$

where η is the center wavelet frequency.

Since $f(t)$ is Lipschitz α at 0, we derive from (6.20) in Theorem 6.4 that equation (6.21) is satisfied for $\alpha' = \alpha$. One can not decrease α' because at the ridge locations, (4.108) proves that

$$|Wf(u,s)| = O(s^{1/2}a(u)) = O(s^{1/2}|u|^\alpha).$$

Exercise 6.7. a) We proved in exercice 6.5 that if $f(t) = |t|^\alpha$ then $Wf(u,s) = s^{\alpha+1/2} Wf(u/s, 1)$. It results that the complex phase $\Phi(u,s)$ of $Wf(u,s)$ satisfies $\Phi(u,s) = \Phi(u/s, 1)$. The lines of constant phase are thus along curves (u,s) which satisfy $u/s = cst$, and which converge to 0 when s goes to 0.
b) A modulus maxima point u_0 is an extrema of $Wf(u,s) = f \star \tilde{\psi}_s(u)$ with $\tilde{\psi}_s(t) = s^{-1/2}\psi(-st)$ and thus satisfies $\partial Wf(u_0, s)/\partial u = 0$. But

$$\frac{\partial Wf(u,s)}{\partial u} = -sW^1 f(u,s) = -sf \star \tilde{\psi}^1_s(u)$$

with $\psi^1(t) = \psi'(t)$. The modulus maxima computed with ψ are thus zeros of a wavelet transform computed with ψ'. Let us now consider the analytic wavelet transform $W^a f(u,s) = f \star \tilde{\psi}^a_s(u)$ computed with the analytic complex wavelet

$$\psi^a(t) = \psi'(t) + iH(\psi')(t)$$

where $H(\psi')$ is the Hilbert transform of ψ'. The modulus maxima of $Wf(u,s)$ correspond to points (u,s) where $W^a f(u,s)$ has a real part equal to zero and hence a phase equal to π. The modulus maxima points thus correspond to points of constant phase (equal to π) on this analytic complex wavelet tranform.

Exercise 6.8. One has

$$Wf(u,s) = \frac{1}{\sqrt{s}} \int_0^{+\infty} \psi\left(\frac{u-t}{s}\right) \mathrm{d}t = \sqrt{s} \int_{-u/s} \psi(t) \mathrm{d}t$$

so that

$$\frac{\partial Wf}{\partial u}(u,s) = 0 \iff \psi(-u/s) = 0.$$

Since ψ is continuous and orthogonal to polynomials of degree less than $p-1$, it has at least p zeros. Indeed, if this were not the case, there would exist less than $p-1$ zero crossings $\{t_k\}_k$ where ψ changes of sign at t_k. Denoting $P(t) = \prod_k (t - t_k)$ which is a degree less than p, $P\psi$ is non zero and of constant sign, so $\langle f, P \rangle \neq 0$, which is a contradiction.

Exercise 6.10. If $f(t) = \int_0^t d\mu_\infty(t)$ is a Cantor devil's staircase then its singularities are located on the support of the Cantor positive measure $d\mu_\infty$. Let t_0 be on the support of $d\mu_\infty$. By construction, explained page 245, for any $p > 0$ there exists an integer q such that $t_0 = q3^{-p}$, and either $(q+1)3^{-p}$ or $(q-1)3^{-p}$ is also on the support of $d\mu_\infty$. Suppose that it is the first case (the second one is treated similarly). The constrution shows that

$$\int_{(q-1)3^{-p}}^{q3^{-p}} d\mu_\infty(t) = 0 \quad , \quad \int_{q3^{-p}}^{(q+1/9)3^{-p}} d\mu_\infty(t) > 0 \quad , \quad \int_{(q+1)3^{-p}}^{(q+2)3^{-p}} d\mu_\infty(t) = 0 \quad . \tag{4}$$

If $\psi = -\theta'$ then according to (6.32)

$$Wf(u,s) = s(f' \star \tilde{\theta}_s)(u) = s \int s^{-1/2} \theta(s^{-1}(t-u)) \, d\mu_\infty(t) \; . \tag{5}$$

Let K be such that θ has a support included in $[-K, K]$. To simplify explanations we suppose that $\theta > 0$ for $|t| < K$. If $s_p^{-1} = K3^{p+1}$ then $\theta(s_p^{-1}(t-u))$ has a support included in $[-3^{-p-1}, 3^{-p-1}]$. It results from (4) and (5) that

$$Wf((q-1/2)3^{-p}, s_p) = 0 \quad , \quad Wf(q3^{-p}, s_p) > 0 \quad , \quad Wf((q+3/2)3^{-p}, s_p) = 0 \quad .$$

So necessarily, $|Wf(u,s_p)|$ has at least one local maxima at position u_p in the interval $[(q-1/2)3^{-p},(q+3/2)3^{-p}]$ and since $t_0 = q3^{-p}$

$$\lim_{p\to+\infty} u_p = t_0 \ .$$

Exercise 6.14. a) One has $\hat\psi(\omega) = \|\omega\|^2\hat\theta(\omega)$ so that

$$F(\omega) = \sum_{j=-\infty}^{+\infty} |\psi(2^j\omega)|^2 = \sum_{j=-\infty}^{+\infty} 2^{4j} e^{-2^{2j+1}\|\omega\|^2}.$$

One has

$$\sum_{j\leqslant 0} |\hat\psi(2^j\omega)|^2 \leqslant \sum_{j\leqslant 0} 2^{4j} = C < +\infty.$$

Using the fact that $2^{j+1} \leqslant 1+(2j+1)\log(2)$, one has

$$\sum_{j>0} |\hat\psi(2^j\omega)|^2 \leqslant \sum_{j>0} e^{4j\log(2)-(1+(2j+1)\log(2))\|\omega\|^2} \leqslant C' \sum_{j<0} e^{j(4\log(2)-2\|\omega\|^2\log(2))}.$$

This shows that for $0 \leqslant \|\omega\|^2 \ \leqslant \alpha < 2$, $F(\omega) \leqslant B$ and also

$$F(\omega) \geqslant \sum_j 2^{4j} e^{-2^{2j+1}\alpha} = A > 0.$$

This proves that for $0 \leqslant \|\omega\|^2 \ \leqslant \alpha$, $A \leqslant F(\omega) \leqslant B$. Using the fact that $F(2\omega) = F(\omega)$, this bound is extended to all $\omega \in \mathbb{R}^2$.
b) For $\psi = \Delta\theta$, where Δ is the Laplacian, one has

$$Wf(u,2^j) = 0 \quad \Longleftrightarrow \quad \Delta(f\star\theta_j) = 0,$$

so this is a multiscale crossing of the Laplacian edge detector.

Suppose that the image can locally be approximated by a straight-edge profile

$$f(x_1,x_2) = \rho(x_1\cos\alpha - x_2\sin\alpha) \tag{6}$$

where $\rho(t)$ is a monotonous function with an inflection point at $t=0$. A direct computation shows that if ψ is the Laplacian of a Gaussian θ then $Wf(u,s)=0$ when u is an inflection point of

$$f\star\theta_s(x) = \rho_s(x_1\cos\alpha - x_2\sin\alpha)$$

which is located along a straight edge curve of angle α, $x_1\cos\alpha - x_2\sin\alpha = t_0$ with $\rho_s''(t_0)=0$. It thus corresponds to the position of modulus maxima point computed with the two partial derivative wavelets ψ^1 and ψ^2.

Suppose that f has a curved edge, which can be modeled with an angle α which has a slow variation as a function of (x_1,x_2). Then the location of the zero-crossings of $Wf(u,s)$ are not identical to the modulus maxima of the first derivative wavelets because of second order terms that are not identical in both wavelet transforms. The zero-crossings are not exactly located at the inflection points of $f\star\theta_s(x)$ but close to these points if the curvature is small.

Exercise 6.15. One has, denoting $\psi_s(t) = \psi(t/s)/\sqrt{s}$,

$$\begin{aligned}
&E(WB(u_1, s)WB(u_2, s)) \\
&= E([B_H \star \bar{\psi}_s(u_1)][B_H \star \bar{\psi}_s(u_2)]) \\
&= \iint E(B_H(t)B_H(t'))\bar{\psi}_s(u_1 - t)\bar{\psi}_s(u_2 - t')\mathrm{d}t\mathrm{d}t' \\
&= \sigma^2 \iint (|t|^{2H} + |t'|^{2H} - |t - t'|^{2H})\bar{\psi}_s(u_1 - t)\bar{\psi}_s(u_2 - t')\mathrm{d}t\mathrm{d}t' \\
&= -\sigma^2 \iint |t - t'|^{2H}\bar{\psi}_s(u_1 - t)\bar{\psi}_s(u_2 - t')\mathrm{d}t\mathrm{d}t'
\end{aligned}$$

where we have use the fact that $\int \psi_s = 0$ to derive the last equality. Performing the successive changes of variables $t - t' \to t$ and $u_2 - t' \to t'$, one gets

$$E(WB(u_1, s)WB(u_2, s)) = -\sigma^2 \int |t|^{2H}\bar{\psi}_s \star \psi_s(u_1 - u_2 - t)\mathrm{d}t$$

which gives the result after the change of variable $t'/s \to t'$ in the convolution.

6 Chapter 7

Exercise 7.1. a) One has

$$\hat{\phi}(\omega) = \prod_{j \geqslant 0} \hat{h}\left(\frac{\omega}{2^{j+1}}\right).$$

For each $j \geqslant 0$ and $\ell \in \mathbb{Z}$, $h(\omega/2^{j+1})$ has a zero of order p at location $2\pi(2^j(2\ell+1))$. Remarking that any $k \in \mathbb{Z}^*$ can be written $k = 2^j(2\ell + 1)$ with $j \geqslant 0$ and $\ell \in \mathbb{Z}$, one sees that $\hat{\phi}(\omega)$ has a zero of order p at each $\omega = 2k\pi$, $k \neq 0$.
b) One has, using Theorem 3.1

$$\begin{aligned}
A(\omega) &= \sum_n n^q \phi(n) e^{in\omega} = \frac{1}{i^q}\frac{d^q}{d\omega^q}\left(\sum_n \phi(n)e^{in\omega}\right) \\
&= \frac{1}{i^q}\frac{d^q}{d\omega^q}\left(\sum_n \hat{\phi}(\omega + 2n\pi)\right) = \frac{1}{i^q}\sum_n \hat{\phi}^{(q)}(\omega + 2n\pi)
\end{aligned}$$

One thus have, for $\omega = 0$,

$$\sum_n n^q \phi(n) = A(0) = \frac{1}{i^q}\sum_n \hat{\phi}^{(q)}(2n\pi) = \frac{1}{i^q}\hat{\phi}^{(q)}(0) = \int t^q \phi(t)\mathrm{d}t,$$

since $\hat{\phi}^{(q)}(2n\pi) = 0$ for $n \neq 0$.

Exercise 7.2. Let $A(t) = \sum_n \phi(t - n)$. Its Fourier transform satisfies, using Theorem 2.4, in the sense of distributions

$$\hat{A}(\omega) = \hat{\phi}(\omega)\sum_n e^{-in\omega} = \hat{\phi}(\omega)2\pi\sum_n \delta(\omega - 2k\pi) = 2\pi\sum_n \phi(2k\pi)\delta(\omega - 2k\pi).$$

Using Exercise 7.1, one has that $\hat{\phi}(2k\pi) = 0$ for $k \neq 0$, and $\hat{\phi}(0) = 1$ so that $\hat{A}(\omega) = 2\pi\delta(\omega)$ and hence $A(t) = 1$.

Exercise 7.3. We choose m to be even so that $\hat{\phi}_m$ is a symmetric function

$$\hat{\phi}_m(\omega)^{-2} = 1 + \sum_{k\neq 0} \frac{1}{(1+2k\pi/\omega)^{2m+2}}.$$

For $0 < \omega \leqslant a < \pi$, one has $|2n\pi/\omega| \geqslant |n|(1+\varepsilon)$ with $0 < \varepsilon < 1$ so that

$$|\hat{\phi}_m(\omega)^{-2}| \leqslant 1 + 2\sum_{n\geqslant 1} \frac{1}{(n(2+\varepsilon)-1)^{(2m+2)}}.$$

We split the sum into $1 \leqslant n \leqslant 1/\varepsilon$ and $1/\varepsilon < n$. For the first part, the sum is finite and one has

$$\sum_{n=1}^{1/\varepsilon} \frac{1}{(n(2+\varepsilon)-1)^{2m+2}} \longrightarrow 0$$

when $m \longrightarrow +\infty$. For the second part, one has

$$\sum_{n>1/\varepsilon} \frac{1}{(n(2+\varepsilon)-1)^{2m+2}} \leqslant \sum_{n>1/\varepsilon} \frac{1}{(2n)^{2m+2}} \leqslant \int_{1/\varepsilon}^{+\infty} \frac{1}{(2x)^{2m+2}} \mathrm{d}x \longrightarrow 0$$

when $m \longrightarrow +\infty$. This shows that $\hat{\phi}_m(\omega) \longrightarrow 1$ for $m \in (-\pi, \pi)$. A similar derivation shows $\hat{\phi}_m(\omega) \longrightarrow 0$ for $|\omega| > \pi$, so that $\hat{\phi}_m \to \hat{\phi}$ almost everywhere. Using dominated convergence and Plancherel formula, this shows that $\|\phi_m - \phi\| \to 0$.

Exercise 7.4. a) If $K = 2$, then $\mathrm{Supp}(\phi) \subset [0,1]$ and

$$\phi(t) = m[0]\phi(2t) + m[1]\phi(2t-1),$$

so that

$$\phi(0.\varepsilon_0\varepsilon_1\cdots\varepsilon_i) = m[0]\phi(\varepsilon_0.\varepsilon_1\cdots\varepsilon_i) + m[1]\phi((\varepsilon_0-1).\varepsilon_1\cdots\varepsilon_i).$$

If $\varepsilon_0 = 1$, then $\phi(\varepsilon_0.\varepsilon_1\cdots\varepsilon_i) = 0$ and if $\varepsilon_0 = 0$ then $\phi((\varepsilon_0-1).\varepsilon_1\cdots\varepsilon_i) = 0$ so that one has

$$\phi(0.\varepsilon_0\varepsilon_1\cdots\varepsilon_i) = m[\varepsilon_0]\phi(0.\varepsilon_1\cdots\varepsilon_i)$$

and by recursion one obtains

$$\phi(0.\varepsilon_0\varepsilon_1\cdots\varepsilon_i) = m[\varepsilon_0]\cdots m[\varepsilon_i]\phi(0).$$

b) If $t = 0.\varepsilon_0\varepsilon_1\ldots$ is a dyadic number, then there exists some K such that $\varepsilon_k = 0$ for $k \geqslant K$. For $m[0] > 1$, if $\phi(0) \neq 0$, this implies that $\phi(t)$ is infinite.
c)

Exercise 7.5. a) The recursion formula is written over the Fourier domain as

$$\hat{\phi}_{k+1}(\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega/2)\hat{\phi}_k(\omega/2)$$

so that, using Plancherel formula

$$a_{k+1}[n] = \frac{1}{4\pi}\int |\hat{h}(\xi/2)|^2|\hat{\phi}_k(\xi/2)|^2 e^{in\xi}\mathrm{d}\xi.$$

Using the Poisson summation formula, Theorem 2.4, one has the following equality, in the sense of distributions

$$\hat{a}_{k+1}(\omega) = \frac{1}{4\pi}\int |\hat{h}(\xi/2)|^2|\hat{\phi}_k(\xi/2)|^2\left(\sum_n e^{-in(\omega-\xi)}\right)\mathrm{d}\xi$$
$$= \frac{1}{2}\sum_n |\hat{h}(\omega/2-n\pi)|^2|\hat{\phi}_k(\omega/2-n\pi)|^2$$

b) If $\phi_k \to \phi$, then $a_k \to a$ where $a[n] = \langle\phi(t),\, \phi(t-n)\rangle$. If $|\hat{h}(\omega)|$ is bounded, then P is a bounded linear operator on $\mathrm{L}^2(\mathbb{R})$. Hence $Pa_k \to Pa$ and thus $Pa = a$, which means that a is an eigenvector of P for the eigenvalue 1.

One has

$$\hat{\phi}_k(\omega) = \prod_{p=1}^{k} 2^{-1/2}\hat{h}(2^{-p}\omega)\hat{\phi}_0(\omega/2^k)$$

Since $\hat{\phi}_0(\omega/2^k) \to 1$ when $k \to +\infty$ and $\hat{\phi}_k$ converges to $\hat{\phi}$, one has

$$\hat{\phi}(\omega) = \prod_{p\geqslant 1} 2^{-1/2}\hat{h}(2^{-p}\omega).$$

Exercise 7.6. a) Since $f \in V_L$, one has

$$f(x) = \sum_k \langle f,\, \phi_{L,k}\rangle\phi_{L,k}$$

$$b[n] = \sum_k a_L[k]2^{-L/2}\phi(k-n)$$

so that $b = 2^{-L/2}a_L \star \phi_d$ where $\phi_d[n] = \phi(n)$.
b) Using Theorem 3.1, one has

$$\hat{\phi}_d(\omega) = \sum_k \hat{\phi}(\omega + 2k\pi).$$

The filter ϕ_d^{-1} whose Fourier transform is $1/\hat{\phi}_d(\omega)$ is stable if $\hat{\phi}_d(\omega) \geqslant A > 0$.
d) If $\{\tilde{\phi}_{L,n}\}_n$ is an interpolation basis of V_L for the points $2^L n$, then

$$f = \sum_n b[n]\tilde{\phi}_{L,n} = \sum_n a_L[n]\phi_{L,n}.$$

The change from b to a_L is stable if and only if the basis $\{\tilde{\phi}_{L,n}\}_n$ is stable.

Exercise 7.7. a) A computation similar to the proof of Theorem 7.11 shows that the reconstruction property is equivalent to having

$$\hat{h}(\omega+\pi)\hat{\tilde{h}}(\omega) + \hat{g}(\omega+\pi)\hat{\tilde{g}}(\omega+\pi) = 0$$

and

$$\hat{h}(\omega)\hat{\tilde{h}}(\omega) + \hat{g}(\omega)\hat{\tilde{g}}(\omega) = 2e^{-il\omega}.$$

One verifies that for a given h, the proposed filters satisfy the first condition, and that the second condition is equivalent to

$$\hat{h}^2(\omega) - \hat{h}^2(\omega + \pi) = 2e^{-il\omega}.$$

b) Denoting $z = e^{-i\omega}$, this last condition is re-written as

$$(\sum_k h[k]z^k)^2 - (\sum_k (-1)^k h[k]z^k)^2 = 2z^l.$$

One verifies that in the left hand side of this equation, all terms of even degrees cancel out, so that ℓ is necessarily odd.
c) For the Haar filter, $\sqrt{2}\hat{h}(\omega) = 1 + e^{-i\omega}$, so that

$$\hat{h}^2(\omega) - \hat{h}^2(\omega + \pi) = (1+z)^2/2 - (1-z)^2/2 = 2z$$

which corresponds to the quadrature mirror filter condition with $\ell = 1$.

Exercise 7.8. A Daubechies wavelet $\psi_{j,n}$ with p vanishing moments has a support of size $2^j(2p-1)$. One thus concentrates on the scales 2^j such that $2^j(2p-1) < \min_k |\tau_k - \tau_{k-1}|$ such that each wavelet intersects only one discontinuity. Since the wavelets are translated by 2^j with support of size $2^j(2p-1)$, each singularity generates exactly $2p-1$ non zero coefficients. One should thus choose $p = q+1$. If $p < q+1$, then there are 2^{-j} coefficients at each scale, that can be all non zero.

Exercise 7.9. a) Denoting $u = 1_{[0,+\infty)}$ and $v = 1_{(1,+\infty)}$, one has $\theta_1 = u - v$. Denoting $f^{(k)}$ the convolution k times of f with itself, one has

$$\theta_m = (u-v)^{(m+1)} = \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} u^{(k)} v^{(m-k)}.$$

One has $u^{(2)} = [t]_+, u^{(3)} = ([t]_+)^2/2$, and more generally, by induction, one proves that

$$u^{(k)}(t) = \frac{([t]_+)^{k-1}}{(k-1)!} \quad \text{and} \quad v^{(k)}(t) = \frac{([t-k]_+)^{k-1}}{(k-1)!}.$$

One then verifies that

$$u^{(i)} \star v^{(j)}(t) = \frac{1}{(i+j+1)!}([t-j]_+)^{i+j+1},$$

so that

$$\theta_m(t) = \frac{1}{m!} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} ([t-j]_+)^m.$$

b) One has, using the Fourier convolution theorem

$$|\theta_m(\omega)| = |\theta_0(\omega)|^{m+1} = |2\sin(\omega/2)/\omega|^{m+1}$$

One has using (7.9)

$$\begin{aligned} B_m &= \max_\omega \sum_k |\hat{\theta}_m(\omega + 2k\pi)|^2 \\ &= \max_\omega 2^m |\sin(\omega/2)|^{2m+2} \sum_k \frac{1}{|\omega + 2k\pi|^{2m+2}}. \end{aligned}$$

By comparison with an integral, one obtains

$$B_m \geqslant |\sin(\omega/2)| \sum_{k \geqslant 0} \frac{1}{|(\omega/(2\pi)+k)\pi|^{2m+2}} \geqslant |\sin(\omega/2)| \int_{\omega/(2\pi)}^{+\infty} \frac{\mathrm{d}x}{(\pi x)^{2m+2}}$$
$$= |\sin(\omega/2)| \frac{(2m+1)/\pi}{(\omega/2)^{2m+1}} \longrightarrow +\infty.$$

when $m \to +\infty$ as long as $2 < \omega < 2\pi$.

Exercise 7.10. This proof is quite technical, and we detail a proof inspired by "Unimodular Wavelets for L^2 and the Hardy Space of H^2" by Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee, and Jin Keun Seo.

If $\{\psi_{j,n}\}_{j,n}$ is an orthogonal basis of $\mathrm{L}^2(\mathbb{R})$, then for all $f \in \mathrm{L}^2(\mathbb{R})$, using Parseval formula

$$\hat{f} = \frac{1}{2\pi} \sum_{j,n} \langle \hat{f}, \hat{\psi}_{j,n} \rangle \hat{\psi}_{j,n}.$$

This corresponds to, for all ξ,

$$\hat{f}(\xi) = \frac{2^j}{2\pi} \sum_{j,n} \int \hat{f}(\omega) \hat{\psi}^*(2^j\omega) \psi(2^j\xi) e^{-ik2^j(\xi-\omega)} \mathrm{d}\omega.$$

One has, using Poisson formula, Theorem 2.4, in the sense of distribution with respect to the variable ω, the equality

$$\sum_n e^{-ik2^j(\xi-\omega)} = \frac{2\pi}{2^j} \sum_k \delta(\omega - \xi + 2^{-j}2k\pi),$$

and thus

$$\hat{f}(\xi) = \sum_{j,n} \hat{f}(\xi - 2^{-j}2k\pi) \hat{\psi}^*(2^j\xi - 2k\pi) \hat{\psi}(2^j\xi)$$

and hence

$$\hat{f}(\xi)(1-\theta(\xi)) = \sum_{j,k \neq 0} \hat{f}(\xi - 2^{-j}2k\pi) \hat{\psi}^*(2^j\xi - 2k\pi) \hat{\psi}(2^j\xi)$$

where

$$\theta(\xi) = \sum_j |\hat{\psi}(2^j\xi)|^2.$$

For $\xi_0 > 0$ (the same derivation applies for $\xi_0 < 0$) and for some $\varepsilon > 0$ small enough, we use $\hat{f} = 1_I$ with $I = [\xi_0 - \varepsilon\pi, \xi_0 + \varepsilon\pi]$, and integrate with respect to ξ to obtain

$$\int_I |\theta - 1| \leqslant \sum_{|k2^{-j}| < \varepsilon} \int_I |\hat{\psi}(2^j\xi)| \, |\hat{\psi}(2^j\xi - 2k\pi)| \mathrm{d}\xi,$$

where we have used the fact that $\hat{f}(\xi - 2^{-j}2k\pi) \neq 0$ on I only if $|k2^{-j}| < \varepsilon$. Using Cauchy-Schwartz, and a change of variable $2^j\xi \to \xi$ one obtains

$$\int_I |\theta - 1| \leqslant 2^{-j} \sum_{2^{-j} < \varepsilon} A_j(\xi) \left(\sum_{|k| < \varepsilon 2^j} B_{j,k}(\xi) \right),$$

where
$$A_j(\xi)^2 = \int_{2^j I} |\hat\psi(\xi)|^2 \mathrm{d}\xi \quad \text{and} \quad B_{j,k}(\xi)^2 = \int_{2^j I} |\hat\psi(\xi + 2k\pi)|^2 \mathrm{d}\xi.$$
We note that, if $|k2^{-j}| < \varepsilon$, then all the intervals $2^j I + 2k\pi$ are included in $2^j[\xi_0 - 3\varepsilon\pi, \xi_0 + 3\varepsilon\pi]$, so that
$$\sum_{|k|<\varepsilon 2^j} B_{j,k}(\xi) \leqslant 2\varepsilon 2^j \int_{2^j(\xi_0-3\varepsilon\pi)}^{2^j(\xi_0+3\varepsilon\pi)} |\hat\psi(\xi)|^2 \mathrm{d}\xi.$$
If fallows that
$$\int_I |\theta - 1| \leqslant 2\varepsilon \sum_{2^{-j}<\varepsilon} \int_{2^j(\xi_0-3\varepsilon\pi)}^{2^j(\xi_0+3\varepsilon\pi)} |\hat\psi(\xi)|^2 \mathrm{d}\xi.$$
If ε is small enough with respect to ξ_0, $\varepsilon < \xi_0/(12\pi)$, then all the intervals $2^j[\xi_0 - 3\varepsilon\pi, \xi_0 + 3\varepsilon\pi]$ are disjoint and thus
$$\frac{1}{|I|} \int_I |\theta - 1| \leqslant \frac{1}{\pi} \int_{(\xi_0-3\varepsilon\pi)/\varepsilon}^{+\infty} |\hat\psi(\xi)|^2 \mathrm{d}\xi.$$
If $\hat\psi$ is continuous, then the left part of this inequality tends to $\theta(\xi_0) - 1$ whereas the right part tends to 0 when $\varepsilon \to 0$. This shows that $\theta(\xi_0) = 0$ for all $\xi_0 \neq 0$. When $\hat\psi \in \mathrm{L}^2(\mathbb{R})$, this results holds only for almost all points $\xi_0 \neq 0$ that are Lebesgue points of $\hat\theta$.

To show that the condition
$$\sum_j |\hat\psi(2^j\omega)|^2 = 1 \tag{7}$$
is not sufficient, let us note that the requirement that $\{\psi(t-n)\}_n$ is an orthogonal system is written
$$\sum_k |\hat\psi(\omega + 2k\pi)|^2 = 1. \tag{8}$$
We note that if $\hat\psi(\omega)$ satisfies (7) and (8), then $\hat\psi(\omega/2)$ still satisfies (7) but not (8) anymore, take for instance
$$\hat\psi(\omega) = 1_{[-2\pi,-\pi]\cup[\pi,2\pi]}.$$

Exercise 7.11. If $\hat\psi = 1_I$, is an indicator function, $\psi_{j,k}$ is an orthogonal basis if it satisfies
$$\sum_j |\hat\psi(2^j\omega)|^2 = 1 \quad \text{and} \quad \sum_k |\hat\psi(\xi + 2k\pi)|^2 = 1.$$
This means that one has the disjoint unions
$$\mathbb{R} = \bigcup_j 2^j I = \bigcup_k (I + 2k\pi).$$
One verifies that this is the case for the set
$$I/\omega = [-32, -28] \cup [-7, -4] \cup [4, 7] \cup [28, 32].$$
where in the following we denote $\omega = \pi/7$. The base approximation space is
$$V_0 = \mathrm{Span}(\psi_{j,n})_{j<0, n\in\mathbb{Z}}.$$

A function $\phi \in V_0$ thus satisfies

$$\mathrm{Supp}(\hat{\phi}) \subset \bigcup_{j<0} (2^j I) \subset J$$

where

$$J/\omega = [-16, -14] \cup [-8, -7] \cup [-4, 4] \cup [7, 8] \cup [14, 16].$$

If $\phi \in V_0$, then the function

$$\theta(\omega) = \sum_k |\hat{\phi}(\omega + 2k\pi)|^2,$$

is supported in $\bigcup_k (J + 2k\pi)$, and one verifies that

$$\left(\bigcup_k (J + 2k\pi) \right) \cap [4\omega, 6\omega] = \emptyset$$

which implies that $\theta(\omega) = 0$ on $[4\omega, 6\omega]$. This means that $\{\phi(t-n)\}_n$ cannot be an orthogonal basis of V_0 because otherwise $\theta(\omega) = 1$ for all ω.

Exercise 7.12. The Coiflet condition is written over the Fourier domain as

$$\forall\, 0 < k < p, \quad \frac{\mathrm{d}^k \phi(\omega)}{\mathrm{d}\omega^k}(0) = 0.$$

One has

$$\sqrt{2}\hat{\phi}(2\omega) = \hat{h}(\omega)\hat{\phi}(\omega),$$

and taking the k^{th} derivative of this expression at $\omega = 0$ leads to the equivalent condition that

$$\forall\, 0 < k < p, \quad \frac{\mathrm{d}^k h(\omega)}{\mathrm{d}\omega^k}(0) = 0,$$

which corresponds to the following conditions on the coefficients of h

$$\sum_n n^k h[n] = 0.$$

Exercise 7.13. Using the same derivation as in Exercise 7.12, the discrete signal ψ_j has p vanishing moments if and only if

$$\forall\, 0 \leqslant k < p, \quad \frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\hat{\psi}_j(0) = 0.$$

Taking k derivative of

$$\hat{\psi}_j(\omega) = \hat{g}(2^{j-L-1}\omega)H(\omega) \quad \text{where} \quad H(\omega) = \prod_{p=0}^{j-L-2} \hat{h}(2^p\omega),$$

shows that this is indeed the case if

$$\forall\, 0 \leqslant k < p, \quad \frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\hat{g}(0) = \frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\hat{h}(\pi) = 0$$

which is equivalent to the continuous wavelet ψ having p vanishing moments, see Theorem 7.4.

Exercise 7.14. a) Since ψ has $p \geqslant 1$ vanishing moments, one can decompose

$$\hat{h}(\omega) = (e^{i\omega} + 1)P(e^{-i\omega})/2$$

where P is a polynomial (see (7.91)), so that

$$\hat{h}_1(\omega) = 2\hat{h}(\omega)/(e^{i\omega} + 1) = P(e^{i\omega})$$

is a polynomial. It is obvious that $2(e^{i\omega} - 1)\hat{g}(\omega)$ is a polynomial in $e^{\pm i\omega}$.
b) One has

$$\hat{\psi}(\omega) = \frac{\hat{g}(\omega/2)}{\sqrt{2}} \prod_{p \geqslant 2} \frac{\hat{h}(\omega/2^p)}{\sqrt{2}}.$$

One has the following equality

$$i\omega = (e^{i\omega} - 1) \prod_{p \geqslant 1} \frac{1 + e^{i\omega/2^j}}{2},$$

and since the Fourier transform of ψ' satisfies $\hat{\psi}'(\omega) = i\omega\hat{\psi}(\omega)$, one can write

$$\hat{\psi}'(\omega) = \frac{\hat{g}_1(\omega/2)}{\sqrt{2}} \prod_{p \geqslant 2} \frac{\hat{h}_1(\omega/2^p)}{\sqrt{2}},$$

with

$$\hat{h}_1(\omega) = \frac{2\hat{h}(\omega)}{e^{i\omega} + 1} \quad \text{and} \quad \hat{g}_1(\omega) = 2(e^{i\omega} - 1)\hat{g}(\omega).$$

c) The derivative coefficients are obtained by replacing (h, g) by (h_1, g_1) in the pyramid algorithm.

Exercise 7.15. a) Denoting $\psi_a^r = \mathrm{Re}(\psi_a)$, and using the result of Exercise 2.4, one has

$$2\hat{\psi}_a^r(\omega) = \hat{\psi}_a(\omega) + \hat{\psi}_a^*(-\omega) = \hat{\psi}(\omega)|\hat{h}(\omega/4 - \pi/2)|^2 + \hat{\psi}(-\omega)^*|\hat{h}(-\omega/4 - \pi/2)|^2$$

Since ψ is a real wavelet, $\hat{\psi}(-\omega)^* = \hat{\psi}(\omega)$ and since h is also real, $|h(-\omega)| = |h(\omega)|$, which leads to

$$2\hat{\psi}_a^r(\omega) = \hat{\psi}(\omega)\left(|\hat{h}(\omega/4 - \pi/2)|^2 + |\hat{h}(\omega/4 + \pi/2)|^2\right) = 2\hat{\psi}(\omega)$$

where we have used the fact that $|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2$ because h is a quadrature filter. This proves that $\mathrm{Re}(\psi_a) = \psi$.
c) Denoting g_1 as the finite filter such that $\hat{g}_1(\omega) = \hat{h}(\omega)|\hat{h}(\omega - \pi/2)|^2$, one has to switch from the ordinary computation of the detail coefficients

$$d_{j+1} = (a_j \star \bar{g}) \downarrow 2$$

where $a \downarrow 2$ is the sub-sampling by 2 operator, to the following computation

$$d_{j+1} = \left([(a_{j-1} \star \bar{g}_1) \downarrow 2] \star \bar{g}\right) \downarrow 2.$$

d) One can compute the coefficients $a_j[n] = \langle f, \phi_{j,n_1}(x_1)\phi_{j,n_2}(x_2)\rangle$ from a_{j-1} by applying filtering by $\bar{h}[n_1]\bar{h}[n_2]$ and subsampling along each direction. Then the detail coefficients $d_j^1[n] = \langle f, \psi_{j,n}^1\rangle$ are computed by filtering/sub-sampling a_{j-2} two times by $\bar{h}$ along the x_2 direction and

then by filtering/sub-sampling by $\bar{g}_1$ and then by $\bar{g}$. Similar computations allow one to obtain the other details coefficients $d_j^k[n] = \langle f,\, \psi_{j,n}^k \rangle$ for $k = 2, 3, 4$.
e) We denote by

$$\psi^{[1]}(x) = \psi(x_1)\phi(x_1), \quad \psi^{[2]}(x) = \phi(x_1)\psi(x_2) \quad \text{and} \quad \psi^{[3]}(x) = \psi(x_1)\psi(x_2)$$

the classical 2D wavelets. One has for a real signal f,

$$\langle f,\, \psi_{j,n}^{[1]} \rangle = \mathrm{Re}(\langle f,\, \psi_{j,n}^1 \rangle), \quad \langle f,\, \psi_{j,n}^{[2]} \rangle = \mathrm{Re}(\langle f,\, \psi_{j,n}^2 \rangle),$$

and

$$\langle f,\, \psi_{j,n}^{[3]} \rangle = \mathrm{Re}(\langle f,\, \psi_{j,n}^3 \rangle) + \mathrm{Re}(\langle f,\, \psi_{j,n}^4 \rangle).$$

This implies that

$$|\langle f,\, \psi_{j,n}^1 \rangle|^2 \geqslant |\langle f,\, \psi_{j,n}^{[1]} \rangle|^2, \quad |\langle f,\, \psi_{j,n}^2 \rangle|^2 \geqslant |\langle f,\, \psi_{j,n}^{[2]} \rangle|^2,$$

and

$$|\langle f,\, \psi_{j,n}^3 \rangle|^2 + |\langle f,\, \psi_{j,n}^4 \rangle|^2 \geqslant \frac{1}{2}(\langle f,\, \psi_{j,n}^3 \rangle + \langle f,\, \psi_{j,n}^4 \rangle)^2 \geqslant \frac{1}{2}|\langle f,\, \psi_{j,n}^{[3]} \rangle|^2.$$

This shows that

$$\sum_{j,n,k} |\langle f,\, \psi_{j,n}^k \rangle|^2 \geqslant \frac{1}{2} \sum_{j,n,k} |\langle f,\, \psi_{j,n}^{[k]} \rangle| = \|f\|^2/2,$$

and hence the frame is redundant.

The reverse inequality is more technical, see [108], Appendix A, for a proof. It uses the fact that if a function ψ satisfies $|\hat{\psi}(\omega)| = O((1 + \|\omega\|^s)$ with $s > 1/2$ and $|\hat{\psi}(\omega)| = O(\|\omega\|^\alpha)$ with $\alpha > 0$, then $\{\psi_{j,n}\}_{j,n}$ is a stable frame of its span. It is easy to show that the almost analytic wavelets $\hat{\psi}^k(\omega)$ satisfy these decay conditions.

Exercise 7.16. a) One notes that $\phi_1 = 1_{[0,1)}$ satisfies the scaling equation

$$\phi_1(t) = \phi_1(2t) + \phi_1(2t-1).$$

whereas $\phi_1(t) = 1_{[0,1)}(t)(2t-1)$ satisfies

$$\phi_2(t) = \frac{1}{2}\left(\phi_2(2t) + \phi_2(2t-1) - \phi_1(2t) - \phi_1(2t-1)\right).$$

One has $\langle \phi_1,\, \phi_2 \rangle = 0$ so that $\{\phi_1(t-n), \phi_2(t-n)\}_n$ is an orthogonal basis of the function that are affine on each interval $[n, n+1)$.
b) One needs to apply the Gram-Schmidt orthogonalization process to

$$\{\phi_1, \phi_2, t^2 1_{[0,1)}(t), t^3 1_{[0,1)}(t)\}$$

to obtain

$$\{\phi_1, \phi_2, \psi_1, \psi_2\}.$$

They correspond to the Legendre polynomials on $[0, 1)$, and

$$\{\phi_1(t-n), \phi_2(t-n), \psi_1(t-n), \psi_2(t-n)\}_{n\in\mathbb{Z}}$$

is an orthogonal basis of the functions that are polynomials of degree 3 on each interval $[n, n+1)$. Since they are orthogonal to ϕ_1, ϕ_2, they are orthogonal to polynomial of degree 1 on their support, and hence they have two vanishing moments (and ψ_2 has 3 vanishing moments).

Exercise 7.17. a) One has

$$\begin{aligned}\int_0^1 \alpha^{\mathrm{repl}}(t)\beta^{\mathrm{repl}}(t)\mathrm{d}t &= \sum_k \int_0^1 \alpha(t-2k)\beta^{\mathrm{repl}}(t)\mathrm{d}t + \sum_k \int_0^1 \alpha(2k-t)\beta^{\mathrm{repl}}(t)\mathrm{d}t \\ &= \sum_k \int_{-2k}^{-2k+1} \alpha(t)\beta^{\mathrm{rep}}(t)\mathrm{d}t + \sum_k \int_{2k-1}^{2k} \alpha(t)\beta^{\mathrm{rep}}(t)\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \alpha(t)\beta^{\mathrm{repl}}(t)\mathrm{d}t \\ &= \sum_k \langle \alpha(t),\, \beta(t-2k)\rangle + \sum_k \langle \alpha(t),\, \beta(2k-t)\rangle = 0.\end{aligned}$$

b) We treat the case the case where $\phi, \tilde{\phi}$ are symmetric about $1/2$ and $\psi, \tilde{\psi}$ are ansi-symmetric about $1/2$, which implies $\psi(1-t) = -\psi(t)$. Since the folded signals are 2-periodic, one only needs to consider, at a scale 2^j with $j < 0$, indexes $0 \leqslant n < 2^{-j+1}$. One thus has, if $(j,n) \neq (j',n')$,

$$\langle \psi_{j,n}(t),\, \tilde{\psi}_{j',n'}(t-2k)\rangle = \langle \psi_{j,n},\, \tilde{\psi}_{j',n'+2^{1-j'}k}\rangle = 0$$

$$\langle \psi_{j,n}(t),\, \tilde{\psi}_{j',n'}(2k-t)\rangle = -\langle \psi_{j,n},\, \tilde{\psi}_{j',n'+(1-2k)2^{-j'}}\rangle = 0$$

which implies, using question a), that

$$\langle \psi_{j,n}^{\mathrm{repl}},\, \psi_{j',n'}^{\mathrm{repl}}\rangle = 0,$$

and similarly with the other inner products with scaling functions.

For a given function $f \in \mathrm{L}^2([0,1])$, we extend it by zero outside $[0,1]$ and decompose it in the wavelet basis of $\mathrm{L}^2(\mathbb{R})$

$$f = \sum_{j \leqslant J,n} \langle f,\, \psi_{j,n}\rangle \tilde{\psi}_{j,n} + \sum_n \langle f,\, \phi_{J,n}\rangle \tilde{\psi}_{J,n}^{\mathrm{repl}}$$

so that for $t \in [0,1]$,

$$f(t) = f^{\mathrm{repl}}(t) = \sum_{j \leqslant J,n} \langle f^{\mathrm{repl}},\, \psi_{j,n}^{\mathrm{repl}}\rangle \tilde{\psi}_{j,n}^{\mathrm{repl}} + \sum_n \langle f^{\mathrm{repl}},\, \phi_{J,n}^{\mathrm{repl}}\rangle \tilde{\phi}_{J,n}^{\mathrm{repl}}.$$

Exercise 7.18. a) In the following, we denote $z = e^{-i\omega}$, and denote

$$\hat{p}(\omega) = P(z) = \frac{R(z)R(z^{-1})}{Q(z)Q(z^-1)}.$$

The perfect reconstruction property reads $P(z) + P(-z) = 2$, so that $P(z) = \sum_{k \in \mathbb{Z}} p_k z^k$ necessarily satisfies $p_{2k} = 0$ for $k \neq 0$. Hence one can decompose

$$P(z) = 1 + z\frac{C(z^2)}{D(z^2)} = \frac{C(z^2) + zD(z^2)}{D(z^2)} = \frac{R(z)R(z^{-1})}{Q(z)Q(z^{-1})}$$

where C and D are polynomials with no root in common. Thus by identification $2D(z^2) = R(z)R(z^{-1}) + R(-z)R(-z^{-1})$, which corresponds to

$$\hat{p}(\omega) = \frac{2|\hat{r}(\omega)|^2}{|\hat{r}(\omega)|^2 + |\hat{r}(\omega+\pi)|^2}$$

where $r(\omega) = R(z)$.
b) The constraint on r is rewritten $R(z^{-1}) = z^{k-1}R(z)$, so that

$$R(z)R(-z^{-1}) = -R(-z)R(z^{-1})$$

and hence

$$\begin{aligned}|\hat{r}(\omega) + \hat{r}(\omega+\pi)|^2 &= (R(z) + R(-z))(R(z^{-1}) + R(-z^{-1})) \\ &= R(z)R(z^{-1}) + R(-z)R(-z^{-1}) = |\hat{r}(\omega)|^2 + |\hat{r}(\omega+\pi)|^2\end{aligned}$$

which implies the factorization

$$\hat{p}(\omega) = \frac{2|\hat{r}(\omega)|^2}{|\hat{r}(\omega) + \hat{r}(\omega+\pi)|^2}$$

c) One can choose

$$\hat{h}(\omega) = \sqrt{2}\frac{(1+z)^5}{(1+z)^5 + (1-z)^5}.$$

One verifies that

$$(1+z)^5 + (1-z)^5 = 2 + 20z^2 + 10z^4$$

whose roots are $\pm i\sqrt{1 \pm \sqrt{5}/5}$.

Exercise 7.19. a) Let h_{new} and $\tilde{h}_{new}$ be defined as

$$h_{new}[n] = (h[n] + h[n-1])/2 \quad \text{and} \quad (\tilde{h}_{new}[n] + \tilde{h}_{new}[n-1])/2 = \tilde{h}[n]$$

so that

$$\hat{h}_{new}(\omega) = \hat{h}(\omega)\frac{1+e^{i\omega}}{2} \quad \text{and} \quad \hat{\tilde{h}}_{new}(\omega) = \hat{\tilde{h}}(\omega)\frac{2}{1+e^{i\omega}}.$$

Sine $\hat{h}_{new}(\omega)\hat{\tilde{h}}_{new}(\omega) = \hat{h}(\omega)\hat{\tilde{h}}(\omega)$, the biorthogonality is conserved.

If h and $\tilde{h}$ have p and $\tilde{p}$ vanishing moments, then

$$\hat{h}(\omega) = (e^{i\omega}+1)^p P(e^{i\omega}) \quad \text{and} \quad \hat{\tilde{h}}(\omega) = (e^{i\omega}+1)^{\tilde{p}} \tilde{P}(e^{i\omega})$$

where P and $\tilde{P}$ are polynomials. This shows that h_{new} has $p+1$ vanishing moments, whereas $\tilde{h}_{new}$ has $\tilde{p}-1$ vanishing moments.
b) One has the following matrix expression for the balancing operations

$$\begin{pmatrix}\hat{h}_{new}(\omega) \\ \hat{\tilde{h}}_{new}(\omega)\end{pmatrix} = \begin{pmatrix}\hat{a}(\omega) & 0 \\ 0 & 1/\hat{a}(\omega)\end{pmatrix}\begin{pmatrix}\hat{h}(\omega) \\ \hat{\tilde{h}}(\omega)\end{pmatrix} \quad \text{where} \quad \hat{a}(\omega) = (1+e^{i\omega})/2.$$

Denoting $A = \hat{a}(\omega)$, and using the following decomposition of a scaling matrix

$$\begin{pmatrix}A & 0 \\ 0 & 1/A\end{pmatrix} = \begin{pmatrix}1 & A-A^2 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ -1/A & 1\end{pmatrix}\begin{pmatrix}1 & A-1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 1 & 1\end{pmatrix},$$

one sees that the balancing operation can be implemented with 4 lifting steps.
c) We start by the original filters

$$h = [1] \quad \text{and} \quad \tilde{h} = [-1, 0, 9, 16, 9, 0, -1]/16,$$

after one step of balancing, one obtains

$$h^1_{new} = [1, 1]/2 \quad \text{and} \quad \tilde{h}^1_{new} = [-1, 1, 8, 8, 1, -1]/8,$$

after one other step of balancing, one obtains

$$h^2_{new} = [-1, 2, 6, 2, -1]/4 \quad \text{and} \quad \tilde{h}^2_{new} = [1, 2, 1]/4,$$

which corresponds to the 5/3 biorthogonal wavelets.

Exercise 7.21. After a simple factorization, applying an odd-length filter of size K such that $h[n] = h[K-n-1]$ requires $K-1$ additions and $(K-1)/2$ multiplications.

The direct application 5/3 filters requires 6 additions and 3 multiplications (times N for a signal of length N), whereas the lifting implementation requires 4 additions and 4 multiplications.

The direct application 9/7 filters requires 14 additions and 7 multiplications (times N for a signal of length N), whereas the lifting implementation requires 8 additions and 4 multiplication.

Exercise 7.22. For an interpolation wavelet, one has $\psi(x) = \phi(2x-1)$ and hence $\hat{h}(\omega) = e^{-i\omega}/2$. One verifies that $\tilde{\phi} = \delta$ and

$$\hat{\tilde{g}}(\omega) = 2\hat{g}(\omega + \pi)e^{-i\omega}$$

satisfy the biorthogonality relations, so that

$$\tilde{\psi}(t) = \sum_{k \in \mathbb{Z}} (-1)^k h[k] \delta\left(t - \frac{k+1}{2}\right).$$

For interpolation of order 4, one has

$$h = [-1, 0, 9, 16, 9, 0, -1]/16.$$

One verifies that

$$\forall q < 4, \quad \sum_k (-1)^k h[k] k^q = 0$$

so that the dual wavelet has 4 vanishing moments.

Exercise 7.23. Denoting ϕ_0 the Daubechies orthogonal wavelet with p vanishing moments, one has $\hat{\phi}(\omega) = |\hat{\phi}_0(\omega)|^2$. Following Y. Meyer in “Wavelets with Compact Support”, one has the following formula

$$|\hat{\phi}_0(\omega)|^2 = 1 - \frac{(2p-1)!}{[(p-1)!]^2 2^{2p-1}} \int_0^\omega \sin^{2p-1}(t) \mathrm{d}t$$

which is an even trigonometric polynomial of order $2p-1$. This can be explicitly computed as

$$|\hat{\phi}_0(\omega)|^2 = \frac{1}{2} + \frac{1}{2} \left(\frac{(2p-1)!}{(p-1)! 4^{p-1}} \right)^2 \sum_{k=1}^{p} \frac{(-1)^{k-1} \cos((2k-1)\omega)}{(p-k)!(p+k-1)!(2k-1)}$$

by expanding $\sin^{2p-1}(t)$. When $p \to +\infty$, one has for $|\omega| \neq \pi/2$,

$$|\hat{\phi}_0(\omega)|^2 \longrightarrow \frac{1}{2} + \frac{2}{\pi} \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{2k-1} \cos((2k-1)\omega),$$

which is the Fourier series expansion of $1_{[-\pi/2, \pi/2]}(\omega)$ for $\omega \in [-\pi, \pi]$. This thus shows that

$$|\hat{\phi}_0(\omega)|^2 \longrightarrow 1_{[-\pi/2, \pi/2]}(\omega),$$

and hence $\phi(t)$ converges in $L^2(\mathbb{R})$ toward $\sin(\pi t)/(\pi t)$.

Exercise 7.24. We follow the proof of Theorem 7.22. For each $t = 2^j n + h$ where $|h| \leqslant 2^j$, we write

$$|f(t) - P_{V_j} f(t)| \leqslant |f(t) - f(t-h)| + |P_{V_j} f(t) - P_{V_j} f(t-h)|.$$

Since f is Lipshitz α regular,

$$|f(t) - f(t-h)| \leqslant C_f |h|^\alpha \leqslant C_f 2^{\alpha j}$$

where C_f is the Lipshitz constant. One also has

$$P_{V_j} f(t) - P_{V_j} f(t-h) = \sum_{n=-\infty}^{\infty} \left(f(2^j(n+1)) - f(2^j n) \right) \theta_{j,h}(t-n)$$

where

$$\theta_{j,h} = \sum_{k=1}^{\infty} \left(\Phi_j(t - h - 2^j k) - \Phi_j(t - 2^j k) \right).$$

As in the proof of the proof of Theorem 7.22, since ϕ has exponential decay,

$$\sum_{n=-\infty}^{+\infty} |\theta_{j,h}(t-n)| \leqslant C_\phi$$

and hence

$$|P_{V_j} f(t) - P_{V_j} f(t-h)| \leqslant C_\phi \max_n |f(2^j(n+1)) - f(2^j n)| \leqslant C_\phi C_f 2^{\alpha j}.$$

Exercise 7.25. Let $\phi(x)$ be a 1D interpolation function. We define the 1D transforms $\tau_0(x) = 0$ and $\tau_1(x) = 2x - 1$. For each set $\{\varepsilon_k\}_{k=1}^p$ of p binary values $\varepsilon_k \in \{0, 1\}$, we define

$$\psi^\varepsilon(x_1, \ldots, x_p) = \prod_{k=1}^{p} \phi(\tau_{\varepsilon_k}(x_k)).$$

where we denote $0 \leqslant \varepsilon < 2^p$ the number whose binary expansion is $\{\varepsilon_k\}_{k=1}^p$. Then ψ^0 is an interpolating scaling function, and $\{\psi^\varepsilon\}_{0<\varepsilon<2^p}$ defines $2^p - 1$ interpolating wavelets that can be used to analyze continuous functions defined on $\mathbb{R}^p$.

Exercise 7.28. We do the derivation here in 1D. The foveated signal $Tf(x)$ is obtained from $f(x)$ by a spatially varying convolution with a filter scaled by t around position t, $g(\cdot/t)/t$, where g is a symmetric smooth low pass function

$$Tf(x) = \int_{-\infty}^{+\infty} f(t) g\left(x/t - 1\right) \frac{\mathrm{d}t}{t}.$$

The operator is written over the wavelet domain as

$$Tf = \sum_{j,j',n,n'} K_{j,j',n,n'} \langle f, \psi_{j,n} \rangle \psi_{j',n'}$$

where

$$K_{j,j',n,n'} = \langle T\psi_{j,n}, \psi_{j',n'} \rangle = \iint \psi_{j,n}(t) \psi_{j',n'}(x) g\left(x/t - 1\right) \frac{\mathrm{d}t}{t} \mathrm{d}x.$$

It is shown in "Wavelet Foveation" by Chang, Mallat and Yap that if g and ψ have compact support, with ψ regular with vanishing moments, then $K_{j,j',n,n'}$ decay fast when $|j-j'|$ or $|n-n'|$ increases. Together with the fact that $K_{j,j',n,n'}$ depends only on $j-j'$, this shows that one can approximate the operator as a diagonal one

$$Tf = \sum_{j,n} \lambda_n \langle f,\, \psi_{j,n}\rangle \psi_{j,n}$$

One can show that if g is of class C^α, then $\lambda_n = O(|n|^{-(\alpha+1)})$.

7 Chapter 8

Exercise 8.1. For $A = 2^{j-L}$, one has that $\{\theta_j[m-nA]\}_n$ is orthonormal if and only if

$$\langle \theta_j[m],\, \theta_j[m-nA]\rangle = \theta_j \star \tilde{\theta}_j[An] = \delta[n].$$

Taking the Fourier transform of this equality, and using exercise 3.20, one has that this is equivalent to

$$A^{-1} \sum_{k=0}^{A-1} |\hat{\theta}_j(A^{-1}(\omega - 2k\pi))|^2 = 1. \tag{9}$$

With a similar derivation, for $B = 2^{j+1-L}$, the families $\{\theta^0_{j+1}[m-Bn]\}_n$ and $\{\theta^1_{j+1}[m-Bn]\}_n$ are orthogonal sets of vectors orthogonal to each other if and only if

$$\begin{cases} B^{-1} \sum_{k=0}^{B-1} \hat{\theta}^0_j(B^{-1}(\omega-2k\pi))\hat{\theta}^1_j(B^{-1}(\omega-2k\pi))^* = 0, \\ B^{-1} \sum_{k=0}^{B-1} |\hat{\theta}^s_j(B^{-1}(\omega-2k\pi))|^2 = 1, \forall\, s = 0,1. \end{cases}$$

Similarly to the proof of Theorem 8.1, one verifies these relationships using (9) and

$$\hat{\theta}^0_{j+1}(\omega) = \hat{\theta}_j(\omega)\hat{h}(A\omega) \quad \text{and} \quad \hat{\theta}^1_{j+1}(\omega) = \hat{\theta}_j(\omega)\hat{g}(A\omega).$$

Exercise 8.3. Each node in the tree has 2^p children. Fixing two quadrature mirror filters (h_0, h_1), for each $0 \leqslant s < 2^p$, one defines the filters

$$h_s[n] = \prod_{i=0}^{p-1} h^{s_i}[n_i]$$

where $(s_i)_i$ is the binary expansion of s. The computation of the wavelet packet coefficients is performed by the following filtering process from the top to the bottom of the tree

$$d^{2^p k+s}_{j+1}[n] = d^k_j \star h_s[2n].$$

The number of scale is $\log_2(N)/d$, the number of nodes per scale is 2^{pj}, the number of coefficients per node is $N/2^{jp}$, so that the number of coefficients per scale is N. The complexity per scale is $O(KN)$ where K is the length of the filter, so that the overall complexity is $O(KN\log(N))$.

Exercise 8.4. There are $\sqrt{N}\log(N)$ horizontal and vertical atoms, so $N\log(N)^2$ atoms in total.

One applies the wavelet packet decomposition algorithm along each of the $\sqrt{N}$ rows in $O(K\sqrt{N}\log(\sqrt{N}))$ operations per row. Then one applies the same process to the column. The overall complexity is thus $O(KN\log(N))$ operations.

Exercise 8.5. a) Denoting $\alpha = (1-i)/2$, one has

$$g_k = \alpha e_k + \bar{\alpha}\bar{e}_k \quad \text{where} \quad e_k[m] = e^{\frac{2i\pi}{N}km}.$$

Thus one has

$$\begin{aligned}\langle g_k,\, g_\ell\rangle &= \langle \alpha e_k,\, \alpha e_\ell\rangle + \langle \alpha e_k,\, \bar{\alpha} e_\ell\rangle + \langle \bar{\alpha} e_k,\, \alpha e_\ell\rangle + \langle \bar{\alpha} e_k,\, \bar{\alpha} e_\ell\rangle \\ &= 2|\alpha|^2\delta[k-\ell] + (\bar{\alpha}^2 + \alpha^2)\delta[k+\ell].\end{aligned}$$

One conclude by noticing that $2|\alpha|^2 = N$ and $\alpha^2 = -\bar{\alpha}^2$.
b) One has

$$\langle f,\, g_k\rangle = \bar{\alpha}\langle f,\, e_k\rangle + \alpha\langle f,\, e_k\rangle = \bar{\alpha}\hat{f}[k] + \alpha\hat{f}[k]$$

Exercise 8.6. One considers the following 2-periodic function

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if} \quad 0 \leqslant t < 1, \\ -f(-t) & \text{if} \quad -1 \leqslant t < 0, \\ f(2-t) & \text{if} \quad 1 \leqslant t < 2, \\ -f(t+2) & \text{if} \quad -2 \leqslant t < -1. \end{cases}$$

Since $\tilde{f}$ is antisymmetric, one can expand orthogonally

$$\tilde{f}(t) = \sum_k b_k \sin(\pi k t/2)$$

and the other symmetries of $\tilde{f}$ with respect to 1 and -1 implies that $b_{2k} = 0$.

Since

$$b_{2k+1} = \langle \tilde{f},\, \sin(\pi(k+1/2)t)\rangle = 2\langle \tilde{f},\, \sin(\pi(k+1/2)t)\rangle$$

one sees that $\{\sqrt{2}\sin(\pi(k+1/2)t)\}_k$ is an orthonormal basis of $\mathrm{L}^2[0,1]$.

For a discrete signal $f \in \mathbb{C}^N$, performing anti-symmetry about $-1/2$ and symmetry about $N-1/2$ and $-N+1/2$ shows that

$$g_k[n] = \sqrt{\frac{2}{N}}\sin\left(\frac{\pi}{N}(k+1/2)(n+1/2)\right)$$

is an orthogonal basis of $\mathbb{C}^N$.

Exercise 8.7. One considers the following 2-periodic function

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if} \quad 0 \leqslant t < 1, \\ -f(-t) & \text{if} \quad -1 \leqslant t < 0. \end{cases}$$

Since $\tilde{f}$ is antisymmetric, one can expand orthogonally

$$\tilde{f}(t) = \sum_k b_k \sin(\pi k t)$$

and since

$$b_k = \langle \tilde{f},\, \sin(\pi k t)\rangle = 2\langle \tilde{f},\, \sin(\pi k t)\rangle$$

one sees that $\{\sqrt{2}\sin(\pi k t)\}_k$ is an orthonormal basis of $\mathrm{L}^2[0,1]$.

For a discrete signal $f \in \mathbb{C}^N$, performing anti-symmetry about $-1/2$ shows that

$$g_k[n] = \sqrt{\frac{2}{N}} \sin\left(\frac{\pi}{N} k(n+1/2)\right)$$

is an orthogonal basis of $\mathbb{C}^N$.

Exercise 8.9. The Meyer wavelet ψ is written $\hat{\psi}(\omega) = g(\omega)e^{i\pi\omega}$. A Meyer wavelet $\psi_{j,n}(x) = 2^{-j/2}\psi(x/2^j - n)$ satisfies

$$\hat{\psi}_{j,n}(\omega) = 2^{j/2} b(2^j\omega)\left(\cos(2\pi(k/1/2)2^j\omega) - i\sin(2\pi(k/1/2)2^j\omega)\right)$$

for a windowing function that verifies some compatibility conditions, so that

$$\langle f,\, \psi_{j,n}\rangle = \langle \hat{f},\, \hat{\psi}_{j,n}\rangle = 2^{j/2}\langle \hat{f},\, g^1_{j,k}\rangle - i2^{j/2}\langle \hat{f},\, g^2_{j,k}\rangle$$

where $\{g^1_{j,k}\}_{j,k}$ is a lapped cosine basis and $\{g^2_{j,k}\}_{j,k}$ is a lapped sine basis.

The Meyer wavelet coefficients can be computed by applying an $O(N\log(N))$ FFT followed by $\log(N)$ lapped transform, and thus the complexity of this algorithm is $O(N\log(N)^2)$. The computation of the filter bank wavelet algorithm with FFT would take $O(N\log(N))$ operation, but would not provide as accurate results.

Exercise 8.12. a) An admissible tree is a an admissible binary tree (called root tree) with a collection of admissible binary trees indexed by the leafs of the tree (called leaf trees).
b) We denote as $C_{j,k}$ the number of admissible double trees with a root tree of depth at most j and with leaf trees of depth at most k. One has

$$C_{j,j} = C^2_{j-1,j} + 1 \geqslant C^2_{j-1,j}$$

and $C_{0,j} = B_j$ where B_k is the number of admissible binary tree, so that

$$C_{j,j} \geqslant (B_j)^{2^{j-1}} \geqslant 2^{(j-1)2^{j-1}}.$$

Similarly to the proof of Theorem 8.2,

$$\log_2(C_{j+1,j+1}) \leqslant 2\log(C_{j,j+1}) + 1/4 \leqslant 2^j \log_2(C_{1,j+1}) + \frac{1}{4}\sum_{i=0}^{j-1} 2^i.$$

Since

$$C_{1,j+1} = B_{j+1} + B^2_{j+1} \leqslant 2^{\frac{5}{4}2^j} + 2^{\frac{5}{2}2^j},$$

one obtains

$$\log_2(C_{j+1,j+1}) \leqslant 2^j \log_2(2^{\frac{5}{4}2^j} + 2^{\frac{5}{2}2^j}) + \frac{1}{4}2^j.$$

8 Chapter 9

Exercise 9.1. a) One has

$$\|f - f_N\|^2 = \|\sum_{m\geqslant N} \langle f,\, g_m\rangle \tilde{g}_m\|^2.$$

Using the frame property, and denoting (A, B) the frame bounds, one has

$$A\sum_{m\geqslant N} |\langle f,\, g_m\rangle|^2 \leqslant \|f - f_N\|^2 \leqslant B\sum_{m\geqslant N} |\langle f,\, g_m\rangle|^2.$$

Hence

$$AC \leqslant \sum_{N=0}^{+\infty} N^{2s-1}\varepsilon_l(N,f) \leqslant BC \quad \text{where} \quad C = \sum_{m\geqslant 0} |\langle f,\, g_m\rangle|^2 \sum_{N\geqslant 0} N^{2s-1}.$$

The following is similar to the proof of Theorem 9.1.
b) We denote

$$\tilde{f}_M = \sum_{k<M} |\langle f,\, g_{m_k}\rangle| \tilde{g}_{m_k},$$

and f_M the best M term approximation. One has, using the frame property

$$\|f - f_M\|^2 \leqslant \|f - \tilde{f}_M\|^2 \leqslant= \| \sum_{k\geqslant M} |\langle f,\, g_{m_k}\rangle| \tilde{g}_{m_k}\|^2$$
$$\leqslant B \sum_{k\geqslant M} |\langle f,\, g_{m_k}\rangle|^2 = O(\sum_{k\geqslant M} k^{-2s}) = O(M^{1-2s}).$$

Exercise 9.4. An M-term approximation of the multi-channel signal is defined as

$$f_\Lambda = (f_{k,\Lambda})_k \quad \text{where} \quad f_{k,\Lambda} = \sum_{m\in\Lambda} \langle f_k,\, g_m\rangle g_m,$$

where Λ is a set of $M = |\Lambda|$ indexes. One has

$$\|f - f_\Lambda\|^2 = \sum_{m\notin\Lambda} A_m \quad \text{where} \quad A_m = \sum_k |\langle f_k,\, g_m\rangle|^2.$$

The best support Λ that minimize the approximation error is thus the one that selects the M largest values of A_m.

Exercise 9.5. One has, for $N = 2^{-j}$

$$f_N(t) = P_{V_j}(f)(t) = f \star \Phi(Nt)$$

where Φ is a smooth low frequency function.

We assume that Φ is smooth and compactly supported in $[-K/2, K/2]$, and that $\int \Phi = 1$, $\|\Phi\|_\infty \leqslant C$. One then checks that the approximation error is localized in $[-K/2, K/2]/N$

$$\int_0^1 |f(t) - f \star \Phi(Nt)|^2 \mathrm{d}t \leqslant 2CKN^{-1}.$$

Exercise 9.6. a) For a smooth function, one has

$$|f(x)| = |\int_0^x f'(x)\mathrm{d}x + f(0)| \leqslant \int_0^x |f'(x)|\mathrm{d}x + |f(0)| \leqslant TV(f) + |f(0)| < +\infty.$$

This result caries over to arbitrary function by approximating by a smooth function.
b) We consider $f(x) = \|x\|^{-\alpha}\phi(x)$ where ϕ is some smooth localizing function, that is 1 inside the disc of radius 1. One has, for $\|x\| \leqslant 1$

$$\nabla f(x) = -\alpha \frac{x}{\|x\|^{\alpha+2}}.$$

Using a polar change of variables,

$$\int \|\nabla f(x)\| \mathrm{d}x = C + \alpha \int \frac{\mathrm{d}x}{\|x\|^{\alpha+1}} = C + 2\pi\alpha \int_0^1 \frac{\mathrm{d}r}{r^\alpha}$$

where C account for the total variation outside the unit disk. One can see that for $0 < \alpha < 1$, f is of bounded variation, but f is not bounded.

Exercise 9.7. Formally, we define

$$g(t) = \int_{\mathbb{R}} H(\omega, t) \mathrm{d}\omega \quad \text{where} \quad H(\omega, t) = (i\omega)^p \hat{f}(\omega) e^{i\omega t}.$$

One has

$$|H(\omega, t)| \leqslant |\omega|^p |\hat{f}(\omega)|$$

Using Cauchy-Schartz inequality, one has

$$\int_{|\omega|>\eta} |\omega|^p |\hat{f}(\omega)| \mathrm{d}\omega \leqslant \left(\int_{|\omega|>\eta} |\omega|^{2(p-s)} \mathrm{d}\omega \right)^{1/2} \|f\|_{Sob(s)}$$

which is bounded if $s > p + 1/2$. So using classical Theorem of derivation under the sign $\int$, this shows that f is C^α and $f^{(\alpha)} = g$.

Exercise 9.8. One can verify numerically that Fourier and orthogonal polynomial approximations behave similarely. An orthogonal polynomial of degree p has p oscillations and is similar to a sinusoid function. The linear N-term approximation with orthogonal polynomial of a C^α function is better that the error produced by a Taylor approximation, and is thus $\|f - f_N\|^2 = O(N^{-2\alpha})$

Exercise 9.9. One has

$$\|f - f_M\|^2 \leqslant \sum_{m=0}^{M} (t_{k+1} - t_k)\Delta = \Delta.$$

We use the fact that any bounded variation function can be decomposed as $f = f_1 - f_2$ where f_1, f_2 are increasing functions. Without loss of generality, we assume $f_1(0) = f_2(0) = 0$ and $f_1(1) = f_2(1) = 1$.

For each function f_i, we build a set $T_i = \{t_k^i = f_i^{-1}(i/M)\}_{k=0}^{M-1}$ of points which guarantees that f_i varies of at most $\Delta = 1/M$ over each $[t_k^i, t_{k+1}^i]$. Defining $\{t_k\}_{k=0}^{2M-1} = T_1 \cup T_2$ guarantees that f varies of less than $\Delta = 1/M$ over each $[t_k, t_{k+1}]$. One thus has $\|f - f_M\|^2 = O(M^{-1})$.

Exercise 9.11. We consider a function f such that $|\langle f, \psi_{j,n} \rangle| = 2^{js}$. For dyadic $N = 2^J$, and ordering wavelet coefficients from coarse to fine scales, $\varepsilon_n[N] = \varepsilon_l[N]$.

For $\alpha < s - 1/2$, one has

$$\sum_{j \leqslant 0} \sum_{n} |\langle f, \psi_{j,n} \rangle|^2 2^{-2j\alpha} = \sum_{j} 2^{-j} 2^{-2\alpha j} 2^{2sj} = \sum_{j \leqslant 0} 2^{2j(s-\alpha-1/2)} < +\infty$$

and Theorem 9.4 shows that f is in W^α.

Exercise 9.12. b) Denoting $|\operatorname{supp}(\psi)| = C = 8$, one has for the CK wavelets whose support intersect a singularity at scale 2^j

$$|\langle f, \psi_{j,n} \rangle| \leqslant \|f\|_\infty \|\psi\|_1 2^{j/2},$$

the other coefficients being zero. One has

$$\varepsilon_l[M] \leqslant 2CK\|f\|_\infty^2\|\psi\|_1^2 M^{-1}.$$

Since there is only KC non-zero coefficients per scale, the non-linear approximation selects all non-zero coefficients corresponding to wavelets whose support intersects a singularity at scale 2^j up to scale $-J = M/(KC)$

$$\varepsilon_n[M] \leqslant \sum_{j\leqslant J} CK\|f\|_\infty^2\|\psi\|_1^2 2^j \leqslant 2CK\|f\|_\infty^2\|\psi\|_1^2 2^J = 2CK\|f\|_\infty^2\|\psi\|_1^2\omega^{-M},$$

where $\omega = 2^{-1/(KC)}$.

Exercise 9.13. a) One has

$$\underset{a\in\mathbb{R}}{\text{argmin}} \sum_{n=\ell}^{k} |f[n]-a|^2 = \frac{1}{\ell-k+1}\sum_{n=\ell}^{k} f[n].$$

b) We call $V_{p,\ell}$ the set of signals supported on $[0,\ldots,\ell]$, that assumes less than p different values. Any signal $f_k \in V_{p,k}$ can be decomposed as

$$f_k = f_\ell + a 1_{[\ell,k]}$$

where $f_\ell \in V_{p-1,\ell-1}$ and $a \in \mathbb{R}$. One thus has

$$\min_{f_k\in V_{p,k}} \|f-f_k\|_{[0,k]}^2 = \min_{\ell\in[0,k-1]}\min_{f\in V_{p,\ell-1}} \|f-f_\ell\|_{[0,\ell-1]}^2 + \min_{a\in\mathbb{R}} \|f-a\|_{[\ell,k]}^2,$$

which gives the result.

An algorithm can computes $\varepsilon_{p,k}$ for increasing p and k, and $\Sigma_{p,k}$ which are the locations of discontinuities in the optimal signal in $V_{p,k}$.

One initializes:

- For all k, $\varepsilon_{1,k} = c_{1,k}$ and $\Sigma_{1,k} = \emptyset$.
- For all p, $\varepsilon_{p,1} = 0$ and $\Sigma_{p,0} = \{0,\ldots,0\}$ (p times).

For $p = 2,\ldots,K$, for $k = 1,\ldots,N-1$, compute

$$\varepsilon_{p,k} = \min_{\ell\in[0,k-1]} \varepsilon_{p-1,\ell} + c_{\ell,k}$$

and denoting ℓ^* such that $\varepsilon_{p,k} = \varepsilon_{p-1,\ell^\star} + c_{\ell^\star,k}$, update

$$\Sigma_{p,k} = \Sigma_{p-1,\ell^\star} \cup \{\ell^\star\}.$$

After running this algorithm, $f_{K,N}$ has discontinuities in $\Sigma_{K,N}$. The main numerical cost is the computation of the $c_{p,k}$, which takes $O(N^2)$ for each k, so the overall complexity is $O(KN^2)$.

Exercise 9.14. a) Performing a first order approximation of the phase near point $2^j n$, and assuming that the amplitude is nearly constant over the support of size 2^j, one gets

$$f(t) \approx a(2^j n)\exp\left(i\phi(2^j n) + i\phi'(2^j n)(t-2^j n)\right).$$

One thus has

$$|\langle f,\, \psi_{j,n}\rangle| \approx |a(2^j n)2^{-j/2}\langle\psi(t/2^j-n),\, \exp(i\phi'(2^j n)(t-2^j n))\rangle|$$

and thus performing the change of variable $t/2^j - n \mapsto t/2^j - n$ in the inner product, one obtains

$$|\langle f,\, \psi_{j,n}\rangle| \approx a(2^j n) 2^{j/2} |\hat{\psi}(2^j \phi'(2^j n))|.$$

b) For $f(t) = \sin(1/t)$, one has $\phi(t) = 1/t$, $a = 1$, $\phi'(t) = -1/t^2$, and thus

$$|\langle f,\, \psi_{j,n}\rangle| \approx 2^{j/2} |\psi(2^{-j} n^{-2})|.$$

The ℓ^p norm of the coefficients reads

$$\sum_{j \leqslant 0, 0 \leqslant n \leqslant 2^{-j}} |\langle f,\, \psi_{j,n}\rangle|^p \approx \sum_{j \leqslant 0, 0 \leqslant n \leqslant 2^{-j}} |2^{jp/2} |\psi(2^{-j} n^{-2})|^p. \tag{10}$$

The wavelet function is close to a band pass filter (Shannon wavelet), so $|\hat{\psi}(\omega)|$ is nearly constant equal to $A > 0$ in some interval $[C_1, C_2]$, and small outside. As an approximation, we can thus considers that for each j the sum in (10) is well approximated by restricting it to indexes n that satisfies

$$C_1 \leqslant 2^{-j} n^{-2} \leqslant C_2 \quad \Longrightarrow \quad C_2^{-1/2} 2^{-j/2} \leqslant n \leqslant C_1^{-1/2} 2^{-j/2}.$$

For each j, the number of elements in the sum (10) is thus approximately $C 2^{-j/2}$ for some constant C, so that

$$\sum_{j \leqslant 0, 0 \leqslant n \leqslant 2^{-j}} |\langle f,\, \psi_{j,n}\rangle|^p \approx CA \sum_{j \leqslant 0} 2^{j/2(p-1)}. \tag{11}$$

This ℓ^p norm is thus finite if and only if $p > 1$.
c) Theorem 9.10 tells us that for all $p < 1$, $\varepsilon_n[M] = O(M^{1-2/p})$, so that $\varepsilon_n[M] = O(M^{-\alpha})$ for any $\alpha < 1$.