Correction of the exercises from the book

*A Wavelet Tour of Signal Processing*

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Abstract
These corrections refer to the 3rd edition of the book *A Wavelet Tour of Signal Processing – The Sparse Way* by Stéphane Mallat, published in December 2008 by Elsevier. If you find mistakes or imprecisions in these corrections, please send an email to Gabriel Peyré (gabriel.peyre@ceremade.dauphine.fr). More information about the book, including how to order it, numerical simulations, and much more, can be find online at wavelet-tour.com.

1 Chapter 2

**Exercise 2.1.** For all \( t \), the function \( \omega \mapsto e^{-i\omega t}f(t) \) is continuous. If \( f \in L^1(\mathbb{R}) \), then for all \( \omega \), \( |e^{-i\omega t}f(t)| \leq |f(t)| \) which is integrable. One can thus apply the theorem of continuity under the integral sign \( \hat{f} \) which proves that \( \hat{f} \) is continuous.

If \( \hat{f} \in L^1(\mathbb{R}) \), using the inverse Fourier formula (2.8) and a similar argument, one proves that \( f \) is continuous.

**Exercise 2.2.** If \( |h| = +\infty \), for all \( A > 0 \) there exists \( B > 0 \) such that \( \int_{-B}^{B} |h| > A \). Taking \( f(x) = 1_{[-A,A]} \text{sign}(h(-x)) \) which is integrable and bounded by 1 shows that

\[
    f \ast h(0) = \int_{-B}^{B} \text{sign}(h(t))h(t)dt > A.
\]

This shows that the operator \( f \mapsto f \ast h \) is not bounded on \( L^\infty \), and thus the filter \( h \) is unstable.

**Exercise 2.3.** Let \( f_u(t) = f(t-u) \), by change of variable \( t-u \mapsto t \), one gets

\[
    \hat{f}_u(\omega) = \int f(t-u)e^{-i\omega t}dt = \int f(t)e^{-i\omega(t+u)}dt = e^{-iu\omega}\hat{f}(\omega).
\]

Let \( f_s(t) = f(t/s) \), with \( s > 0 \), by change of variable \( t/s \mapsto t \), one get

\[
    \hat{f}_s(\omega) = \int f(t/s)e^{-i\omega t}dt = \int f(t)e^{-i\omega st}|s|dt = |s|\hat{f}(s\omega).
\]

1
Let \( f \) by \( C^1 \) and \( g = f' \), the by integration by parts, since \( f(t) \to 0 \) where \( |t| \to +\infty \),
\[
\hat{g}(\omega) = \int f'(t)e^{-i\omega t} dt = -\int f(t)(-i\omega)e^{-i\omega t} dt = (i\omega)\hat{f}(\omega).
\]

Exercise 2.4. One has
\[
f_r(t) = \text{Re}[f(t)] = \frac{[f(t) + f^*(t)]}{2} \quad \text{and} \quad f_i(t) = \text{Ima}[f(t)] = \frac{[f(t) - f^*(t)]}{2}
\]
so that
\[
\hat{f}_r(\omega) = \int \frac{f(t) + f^*(t)}{2} e^{-i\omega t} dt = \hat{f}(\omega)/2 + \text{Conj}\left(\int f(t)e^{i\omega t} dt\right)/2
\]
\[
= [\hat{f}(\omega) + \hat{f}^*(-\omega)]/2,
\]
where \( \text{Conj}(a) = a^* \) is the complex conjugate. The same computation leads to
\[
\hat{f}_i(\omega) = [\hat{f}(\omega) - \hat{f}^*(-\omega)]/2.
\]

Exercise 2.5. One has
\[
\hat{f}(0) = \int f(t) dt = 0.
\]
If \( f \in L^1(\mathbb{R}) \), one can apply the theorem of derivation under the integral sign \( \int \) and get
\[
\frac{d}{d\omega} \hat{f}(\omega) = \int -itf(t)e^{-i\omega t} dt \implies \hat{f}'(0) = -i\int tf(t) dt = 0.
\]

Exercise 2.6. If \( f = 1_{[-\pi,\pi]} \) then one can verify that
\[
\hat{f}(\omega) = \frac{2\sin(\pi\omega)}{\omega}.
\]
It result that
\[
\int \frac{\sin(\pi\omega)}{\pi\omega} \frac{d\omega}{\omega} = \frac{1}{2\pi} \int \hat{f}(\omega) d\omega = f(0) = 1.
\]
If \( g = 1_{[-1,1]} \) then \( \hat{g}(\omega)/2 = \sin(\omega)/\omega \). The inverse Fourier transform of \( \hat{g}(\omega)^3 \) is \( g * g * g(t) \) so
\[
\int \frac{\sin^3(\omega)}{\omega^3} d\omega = \frac{1}{8} \int \hat{g}(\omega)^3 d\omega = \frac{2\pi}{8} g * g * g(0) = \frac{3\pi}{4},
\]
where we used the fact that
\[
g * g * g(0) = \int_{-1}^{1} h(t) dt = 3
\]
where \( h \) is a piecewise linear hat function with \( h(0) = 2 \).

Exercise 2.7. Writing \( u = a - ib \), and differentiating under the integral sign \( \int \), one has
\[
f'(\omega) = \int -it e^{-ut^2} e^{-i\omega t} dt.
\]
By integration by parts, one gets an ordinary differential equation
\[
f'(\omega) = \frac{-\omega}{2u} \hat{f}(\omega)
\]
whose solution is
\[ f(\omega) = Ke^{-\frac{1}{2}K^2} \]
for some constant \( K = \hat{f}(0) \). Using a switch from Euclidean coordinates to polar coordinates \((x, y) \to (r, \theta)\) which satisfies \(dxdy = rdr\theta\), one gets
\[
K^2 = \int e^{-ux^2}dx \int e^{-uy^2}dy = \iint e^{-u(x^2+y^2)}dxdy = 2\pi \int_0^{\infty} e^{-ur^2}rdr = \frac{\pi}{u},
\]
which gives the result.

**Exercise 2.8.** If \( f \) is \( C^1 \) with a compact support, with an integration by parts we get
\[
\hat{f}(\omega) = \frac{1}{i\omega} \int f'(t)e^{-i\omega t}dt
\]
so that
\[
|\hat{f}(\omega)| \leq \frac{C}{\omega} \quad \text{with} \quad C = \int |f'(t)|dt < +\infty,
\]
which proves that \( f(\omega) \to 0 \) when \( |\omega| \to +\infty \).

Let \( f \in L^1(\mathbb{R}) \) and \( \varepsilon > 0 \). Since \( C^1 \) functions are dense in \( L^1(\mathbb{R}) \), one can find \( g \) such that
\[
\int |f - g| \leq \varepsilon/2.
\]
Since \( \hat{g}(\omega) \to 0 \) when \( |\omega| \to +\infty \), there exists \( A \) such that \( |\hat{g}(\omega)| \leq \varepsilon/2 \) when \( |\omega| > A \). Moreover, the Fourier integral definition implies that
\[
|\hat{f}(\omega) - \hat{g}(\omega)| \leq \int |f(t) - g(t)| dt
\]
so for all \( |\omega| > A \) we have \( |\hat{f}(\omega)| \leq \varepsilon \) which proves that \( f(\omega) \to 0 \) when \( |\omega| \to +\infty \).

**Exercise 2.9.**

a) For \( f_0(t) = 1_{[0, +\infty)}(t)e^{pt} \), we get
\[
\hat{f}_0(\omega) = \int_0^{+\infty} e^{(p-i\omega)t}dt = \frac{1}{i\omega - p}.
\]
For \( f_n(t) = t^n1_{[0, +\infty)}(t)e^{pt} \), an integration by parts gives
\[
\hat{f}_n(\omega) = \int_0^{+\infty} t^n e^{(p-i\omega)t}dt = \frac{n}{i\omega - p} \hat{f}_{n-1}(\omega),
\]
so that
\[
\hat{f}_n(\omega) = \frac{n!}{(i\omega - p)^n}.
\]

b) Computing the Fourier transform on both sides of the differential equation gives
\[
g = f \ast h \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k (i\omega)^k}{\sum_{k=0}^{M} b_k (i\omega)^k}.
\]
We denote by \( \{p_k\}_{k=0}^L \) the poles of the polynomial \( \sum_{k=0}^M b_k z^k \), with multiplicity \( n_k \). If \( K < M \),
one can decompose the rational fraction into
\[
\hat{h}(\omega) = \sum_{k=0}^L \frac{Q_k(i\omega)}{(i\omega - p_k)^{n_k}}.
\]
where each $Q_k$ is a polynomial of degree strictly smaller than $n_k$. It results that $h(t)$ is a sum of derivatives up to a degree strictly smaller than $n_k$ of the inverse Fourier transform of

$$
\hat{f}_{p_k,n_k}(\omega) = \frac{1}{(i\omega - p_k)^{n_k}}
$$

which is

$$
f_{p_k,n_k}(t) = \frac{1}{n_k!} t^{n_k} 1_{[0, +\infty)}(t) e^{p_k t}.
$$

Each filter $f_{p_k,n_k}$ is causal, stable and $n_k$ times differentiable. It results that that $h$ is causal and stable.

If, there exists $l$ with Re$(p_l) = 0$ then for the frequency $\omega = -ip_l$ we have $|\hat{h}(\omega)| = +\infty$ so $h$ can not be stable.

If, there exists $l$ with Re$(p_l) > 0$ then by observing that $\hat{f}_{p_l,n_l}(-\omega) = (-1)^{n_l}(i\omega + p_l)^{-n_l}$ and by applying the result in a) we get

$$
f_{p_l,n_l}(t) = \frac{1}{n_l!} t^{n_l} 1_{(-\infty,0]}(t) e^{p_l t}
$$

which is anticausal. We thus derive that $h$ is not causal.

c) Denoting $\alpha = e^{i\pi/3}$, one can write

$$
|\hat{h}(\omega)|^2 = \frac{1}{1 - (i\omega/\omega_0)^6}
$$

with

$$
1/\hat{h}(\omega) = (i\omega/\omega_0 + 1)(i\omega/\omega_0 + \alpha)(i\omega/\omega_0 + \alpha^*) = P(i\omega).
$$

Since the zeros of $P(z)$ have all a strictly negative real part, $h$ is stable and causal. To compute $h(t)$ we decompose

$$
\hat{h}(\omega) = \frac{a_1}{i\omega/\omega_0 + 1} + \frac{a_2}{i\omega/\omega_0 + \alpha} + \frac{a_3}{i\omega/\omega_0 + \alpha^*},
$$

we compute $a_1$, $a_2$ and $a_3$ and by applying the result in (a) we derive that

$$
\hat{h}(t) = \omega_0(a_1 1_{[0, +\infty)}(t) e^{-i\omega_0 t} + a_2 1_{[0, +\infty)}(t) e^{-i\alpha \omega_0 t} + a_3 1_{[0, +\infty)}(t) e^{-i\alpha^* \omega_0 t}).
$$

**Exercise 2.10.** For $a > 0$ and $u > 0$ and $g$ a Gaussian function, define

$$
f_{a,u}(t) = e^{-at} g(t - u) + e^{-at} g(t + u).
$$

We verify that $\sigma_a(f_{a,u})$ increases proportionally to $u$. Its Fourier transform is

$$
\hat{f}_{a,u}(\omega) = e^{-i\omega a} \hat{g}(\omega - a) + e^{i\omega a} \hat{g}(\omega + a)
$$

so $\sigma_a(f_{a,u})$ increases proportionally to $a$. For $a$ and $u$ sufficiently large we get the the result.

**Exercise 2.11.** Since $f(t) \geq 0$

$$
|\hat{f}(\omega)| = |\int f(t) e^{-i\omega t} dt| \leq \int f(t) dt = \hat{f}(0).
$$

**Exercise 2.12.** a) Denoting $u(t) = |\sin(t)|$, one has $g(t) = u(t) u(\omega_0 t)$ so that

$$
\hat{g}(\omega) = \frac{1}{2\pi} \hat{a}(\omega) * \hat{u}(\omega/\omega_0)
$$
where \( \hat{u}(\omega) \) is a distribution
\[
\hat{u}(\omega) = \sum_n c_n \delta(\omega - n)
\]
and \( c_n \) is the Fourier coefficient
\[
c_n = \int_{-\pi}^{\pi} |\sin(t)|e^{-int}dt = -\int_{-\pi}^{0} \sin(t)e^{-int}dt + \int_{0}^{\pi} \sin(t)e^{-int}dt.
\]
The change of variable \( t \to t + \pi \) in the first integral shows that \( c_{2k+1} = 0 \) and for \( n = 2k, \)
\[
c_{2k} = 2 \int_{0}^{\pi} \sin(t)e^{-i2kt}dt = \frac{4}{1 - 4k^2}.
\]
One thus has
\[
\hat{u}(\omega) = \frac{1}{2\pi} \sum_n c_n \hat{u}(\omega - n\omega_0) = \frac{2}{\pi} \sum_k \hat{a}(\omega - 2k\omega_0) \frac{1}{1 - 4k^2}.
\]

b) If \( \hat{a}(\omega) = 0 \) for \( |\omega| > \omega_0, \) then \( h \) defined by \( \hat{h}(\omega) = \frac{\pi}{2} \chi_{[-\omega_0,\omega_0]}(\omega) \) guarantees that \( \hat{g} \hat{h} = \hat{a} \) and hence \( a = g \ast h. \)

Exercise 2.13. One has
\[
\hat{g}(\omega) = \frac{1}{2} \sum_n \hat{f}_n(\omega) \ast [\delta(\omega - 2n\omega_0) + \delta(\omega + 2n\omega_0)] = \frac{1}{2} \sum_n [\hat{f}_n(\omega - 2n\omega_0) + \hat{f}_n(\omega + 2n\omega_0)].
\]
Each \( \hat{f}_n(\omega \pm 2n\omega_0) \) is supported in \( [(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0], \) and thus \( \hat{g} \) is supported in \( [-2N\omega_0, 2N\omega_0]. \)

Since the intervals \( [(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0] \) are disjoint, one has
\[
\hat{f}_n(\omega \pm 2n\omega_0) = 2\hat{g}(\omega) \chi_{[-(1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]}(\omega).
\]
The change of variable \( \omega \pm 2n\omega_0 \to \omega \) and summing for \( n \) and \( -n \) gives
\[
\hat{f}_n(\omega) = [\hat{g}(\omega - 2n\omega_0) + \hat{g}(\omega + 2n\omega_0)] \hat{h}(\omega),
\]
where \( \hat{h}(\omega) = \chi_{[-\omega_0,\omega_0]}(\omega) \). Denoting \( g_n(t) = 2g(t)\cos(2n\omega_0t), \) one sees that \( f_n \) is recovered as
\[
f_n = g_n \ast h.
\]

Exercise 2.14. The function \( \phi(t) = \sin(\pi t)/(\pi t) \) is monotone on \([-3/2, 0]\) and \([0, 3/2]\) on which is variation is \( 1 + \frac{2}{\pi^2} \). For each \( k \in \mathbb{N}^* \), it is also monotone on each interval \([k + 1/2, k + 3/2]\) on which the variation is \( \frac{1}{\pi}[(k + 1/2)^{-1} + (k + 3/2)^{-1}] \). One thus has
\[
\|\phi\|_V = 2(1 + \frac{2}{3\pi}) + \frac{2}{\pi} \sum_{k \geq 1} [(k + 1/2)^{-1} + (k + 3/2)^{-1}] = +\infty.
\]
For \( \phi = \lambda 1_{[a,b]}, \) \( |\phi'| = \lambda \delta_a + \lambda \delta_b \) and hence \( \|\phi\|_V = 2\lambda. \)

Exercise 2.16. Let
\[
f(x) = 1_{[0,1]^2}(x_1, x_2) = f_0(x_1)f_0(x_2) \quad \text{where} \quad f_0(x_1) = 1_{[0,1]}(x_1).
\]
One has
\[ f(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \frac{(e^{i\omega_1} - 1)(e^{i\omega_2} - 1)}{\omega_1\omega_2}. \]

Let
\[ f(x) = e^{-x_1^2-x_2^2} = f_0(x_1)f_0(x_2) \quad \text{where} \quad f_0(x_1) = e^{-x_1^2}. \]

One has
\[ \hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \pi e^{-(\omega_1^2+\omega_2^2)/4}. \]

Exercise 2.17. If |t| > 1, the ray \( \Delta_{t, \theta} \) does not intersect the unit disc, and thus \( p_0(t) = 0 \). For \(|t| < 1\), the Radon transform is computed as the length of a cross section of a disc
\[ p_0(t) = 2\sqrt{1-t^2}. \]

Exercise 2.18. We prove that the Gibbs oscillation amplitude is independent of the angle \( \theta \) and is equal to a one-dimensional Gibbs oscillation. Let us decompose \( f(x) \) into a continuous part \( f_0(x) \) and a discontinuity of constant amplitude \( A \):
\[ f(x) = f_0(x) + A u(\cos(\theta)x_1 + \sin(\theta)x_2) \]
where \( u(t) = 1_{[0, +\infty)}(t) \) is the one-dimensional Heaviside function. The filter satisfies \( h_\xi(x_1, x_2) = g_\xi(x_1)g_\xi(x_2) \) with \( g_\xi(t) = \sin(\xi t)/(\pi t) \). The Gibbs phenomena is produced by the discontinuity corresponding to the Heaviside function so we can consider that \( f_0 = 0 \). Let us suppose that \(|\theta| \leq \pi/4\), with no loss of generality. We first prove that
\[ f \ast h_\xi(x) = f \ast g_\xi(x) \quad (1) \]
where \( \hat{g}_\xi(\omega_1, \omega_2) = 1_{[-\xi, \xi]}(\omega_2) \). Indeed \( f(x) \) is constant along any line of angle \( \theta \), one can thus verify that its Fourier transform has a support located on the line in the Fourier plane, of angle \( \theta + \pi/2 \) which goes through 0. It results that \( \hat{f}(\omega)\hat{h}_\xi(\omega) = \hat{f}(\omega)\hat{g}_\xi(\omega) \) because the filtering limits the support of \( \hat{f} \) to \(|\omega_2| \leq \xi \). But \( g_\xi(x_1, x_2) = \delta(x_1) \sin(\xi x_2)/(\pi x_2) \). The convolution (1) is thus a one-dimensional convolution along the \( x_2 \) variable, which is computed in the Gibbs Theorem 2.8. The resulting one-dimensional Gibbs oscillations are of the order of \( A \times 0.045 \).

2 Chapter 3

Exercise 3.1. One has \( \phi_{s, n}(t) = s^{-1/2}1_{[ns, (n+1)s)} \), which satisfies \( \|\phi_{s, n}\| = 1 \) and \( \langle \phi_{s, n}, \phi_{s, n'} \rangle = 0 \) for \( n \neq n' \) because \([ns, (n+1)s) \) and \([n's, (n'+1)s) \) are disjoint. If \( f(x) = a_n \) on each interval \([ns, (n+1)s) \), then
\[ f(x) = \sum_n a_n 1_{[ns, (n+1)s)} = \sum_n \langle f, \phi_{s, n} \rangle \phi_{s, n} \]
So \( \{\phi_{s, n}\}_n \) is an orthonormal basis of functions that are piecewise constant on each interval \([ns, (n+1)s) \).

Exercise 3.2. If \( \text{Supp}(\hat{f}) \subset [-\pi/s, \pi/s] \), then
\[ \hat{f}(\omega) = \hat{f}(\omega)1_{[-\pi/s, \pi/s]}(\omega) = \frac{1}{s}\hat{f}(\omega)\hat{\phi}_s(\omega) \]
and hence using the Fourier convolution theorem and the fact that $\phi_s$ is symmetric,

$$f(u) = \frac{1}{s} f \ast \phi_s(u) = \frac{1}{s} \langle f(t), \phi_s(t-u) \rangle.$$  

**Exercise 3.3. a)** The function $f$ is an interpolation function if and only if

$$\sum_n f(n) \delta(t-n) = \delta(t)$$

in the distribution sense. Using the sampling Theorem 3.1 with $s = 1$, one gets equivalently the equality of the Fourier transform

$$\sum_k \hat{f}(\omega + 2k\pi) = 1.$$  

**b)** One has

$$\hat{f}(\omega) = \sum_n h[n] \theta(\omega)e^{-in\omega} = h(\omega)\theta(\omega).$$

and thus

$$\sum_k \hat{f}(\omega + 2k\pi) = h(\omega)A(\omega) \quad \text{where} \quad A(\omega) = \sum_k \theta(\omega + 2k\pi)$$

If for all $\omega$, $A(\omega) \neq 0$, one can set $h(\omega) = 1/A(\omega)$ and $f(\omega) = \theta(\omega)/A(\omega)$. Thus $f \in L^2(\mathbb{R})$ if it exists $B > 0$ such that $|A(\omega)| > B$, in which case $\|f\| \leq |\theta|/B$.

**Exercise 3.4.** Let $A(t) = \sum_n f(t-n)$, then in the sense of distribution

$$\hat{A}(\omega) = \hat{f}(\omega) \sum_n e^{-in\omega} = \hat{f}(\omega) \sum_n \delta(\omega - 2n\pi) = \sum_n \hat{f}(2n\pi) \delta(\omega - 2n\pi).$$

Taking the inverse Fourier transform leads to

$$\sum_n f(t-n) = \sum_n \hat{f}(2n\pi) e^{2in\pi\omega}.$$  

**Exercise 3.5.** The orthogonal projection of $f$ on $U_s$ is defined by

$$\forall g \in U_s, \quad \langle \tilde{f} - f, \tilde{f} - g \rangle = 0.$$  

It thus satisfies

$$\forall g \in U_s, \quad |f - g|^2 = |f - \tilde{f}|^2 + |\tilde{f} - g|^2 \geq |f - \tilde{f}|^2$$

and hence $\tilde{f}$ minimizes $|f - \tilde{f}|$ subject to $\tilde{f} \in U_s$.

**Exercise 3.6.** The sufficient condition comes from

$$\|Lf\|_\infty \leq |f|_\infty \|h\|_1.$$  

If $\|h\|_\infty = +\infty$, then for any $A > 0$ it exists $B$ such that $\sum_{|k|<B} |h[k]| > A$. Taking $f[k] = \text{sign}(h[-k])1_{[-B,B]}[-k]$ that satisfies $|f|_\infty \leq 1$ shows that

$$Lf[0] = \sum_{|k| \leq B} |h[k]| > A.$$
The operator \( f \mapsto f \ast h \) is not bounded on \( \ell^\infty \), so that the filter is unbounded.

**Exercise 3.7.** One has
\[
\hat{g} \ast \hat{h}(\omega) = \int_{-\pi}^{\pi} \left( \sum_n h[n] e^{-i\xi n} \right) \left( \sum_p g[p] e^{-i(\omega - \xi - \xi(n-p))} \right) d\xi \\
= \int_{-\pi}^{\pi} \sum_{n,p} h[n] g[p] e^{-i\xi(n-p)} e^{-i\omega p} d\xi.
\]

Exchanging signs \( \int \) and \( \sum \), and using the fact that
\[
\int_{-\pi}^{\pi} e^{-i\xi(n-p)} d\xi = \delta[n-p]
\]
shows that
\[
\hat{g} \ast \hat{h}(\omega) = \sum_n h[n] g[n] e^{-i\omega n} = \hat{f}(\omega).
\]

**Exercise 3.8.** Let \( e_k[n] = e^{\frac{2\pi}{s}kn} = \omega^n \) where \( \omega = e^{\frac{2\pi}{s}} \). If \( k \neq k' \), one has a geometrical sum
\[
\langle e_k, e_{k'} \rangle = \sum_{n} \omega^{kn} \omega^{-k'n} = \sum_{n} (\omega^{k-k'})^n = \frac{1 - \omega^{n(k-k')}}{1 - \omega^{k-k'}} = 0
\]
because \( \omega^N = 1 \). Since \( \|e_k\| = \sqrt{N} \), the family \( \{e_k/\sqrt{N}\}_k \) is an orthonormal basis of \( \mathbb{C}^N \).

**Exercise 3.9.** Denoting
\[
f_d(t) = \sum_k f(ks) \delta(t - ks),
\]
one has using Theorem 3.1
\[
\hat{f}_d(\omega) = \frac{1}{s} \sum_k \hat{f}(\omega - 2k\pi/s).
\]
Since \( \hat{f} \) is supported in \( I_n = [-n\pi/s, -n\pi/s] \cup [n\pi/s, (n+1)\pi/s] \), and the intervals \( I_n + 2k\pi \) are disjoint, one has
\[
\hat{f}(\omega) = \hat{f}_d(\omega) \hat{\phi}_s(\omega) \quad \text{where} \quad \hat{\phi}_s(\omega) = s1_{I_n}(\omega),
\]
which corresponds to the reconstruction formula
\[
f(t) = \sum_n f(ns) \phi_s(t - ns)
\]
with the kernel obtained by inverse Fourier transform formula
\[
\phi_s(t) = s \int_{-n\pi/s}^{-(n+1)\pi/s} e^{i\omega t} d\omega + s \int_{n\pi/s}^{(n+1)\pi/s} e^{i\omega t} d\omega \\
= \frac{1}{\pi\omega/s} [\sin((n+1)\pi/s) - \sin(n\pi/s)]
\]

**Exercise 3.10.** a) One has \( \hat{f}(ns) = f \ast \phi_s \) where \( \phi_s = 1_{[-s/2, s/2]} \).

b) Since \( \hat{\tilde{f}}(\omega) = \hat{f}(\omega) \hat{\phi}_s(\omega) \), \( \text{supp}(f) \subset [-\pi/s, \pi/s] \), and hence \( \tilde{f} \) is recovered using Shannon interpolation formula.
c) One has
\[ \hat{\tilde{f}}(\omega) = \hat{f}(\omega) \frac{\sin(\omega s/2)}{\omega s/2} \]
and thus
\[ f = \tilde{f} \star \psi_s \quad \text{with} \quad \hat{\psi}_s(\omega) = \frac{\omega s/2}{\sin(\omega s/2)}. \]

d) For \( \omega \in [-\pi/s, \pi/s] \), one has
\[ \hat{\phi}_s(\omega) > \frac{2}{\pi} \]
and hence \( ||f|| \leq \frac{\pi}{2} ||\tilde{f}|| \) which shows that the reconstruction is stable.

Exercise 3.11. a) \( \phi \) is supported in \([-1, 1]\), on \( t \in [-1, 0] \), \( \phi(t) = t + 1 \), on \( t \in [0, 1] \), \( \phi(t) = 1 - t \).

b) If \( f(t) \) is linear on each interval \([n, n+1]\), then
\[ f(t) = \sum_n f(n) \phi(t-n). \]

One has, using (7.21),
\[ \sum_k |\hat{\phi}(\omega - 2k\pi)|^2 = \sum_k \frac{\sin^4(\omega/2)}{1 + 2 \cos^2(\omega/2)} = \frac{1}{3} (1 + 2 \cos^2(\omega/2)) \geq \frac{1}{3}, \]
so using Theorem 3.4, \( \{\phi(t-n)\}_n \) is a Riesz basis of the space of piecewise linear functions on each interval \([n, n+1]\).

c) The dual basis satisfies
\[ \tilde{\phi}(\omega) = \frac{3 \sin^2(\omega/2)}{(\omega/2)^2 (1 + 2 \cos^2(\omega/2))} = \frac{\hat{h}(\omega)}{-\omega^2} \]
where \( \hat{h}(om) = -12 \sin^2(\omega/2)/(1 + 2 \cos^2(\omega/2)) \) is the Fourier series of a discrete filter. It results that \( \phi(t) \) is obtained by integrating twice \( h(t) = \sum_n h[n] \delta(t-n) \). The Fourier series \( \hat{h}(\omega) \) is a rational fraction of \( e^{-i\omega} \) which is not reducible to a polynomial so \( h[n] \) has an infinite support, which proves that \( \phi(t) \) also has an infinite support.

Exercise 3.12. One has
\[ |\hat{f}[k]| = \left| \sum_n f[n] e^{-2\pi i n k} \right| \leq \sum_n |f[n]| |e^{-2\pi i n k}| = \sum_n |f[n]|. \]

Exercise 3.13. a) Let \( h[0] = 1, h[-1] = -1 \) and \( h[n] = 0 \) for \( n \not\in \{-1, 0\} \). Then
\[ |f|_V = \sum_n |f \otimes h[n]| \]
and
\[ \hat{h}[k] = 1 - e^{2\pi i k} \quad \Rightarrow \quad |h[k]| = 2|\sin(k\pi/N)|. \]

b) By applying the result of the exercise 3.12 we have that the Fourier transform \( \hat{f}[k] \hat{h}[k] \) of \( f \otimes h[n] \) satisfies
\[ |\hat{f}[k] \hat{h}[k]| \leq |f|_V \]
so for $|k| \leq N/2$ we verify that

$$|\hat{f}[k]| \leq \frac{|f|_V}{2|\sin(k\pi/N)|} \leq \frac{|f|_V}{2k} \leq \frac{N|f|_V}{2k}$$

since $\sin(x) \geq 2x/\pi$ for $|x| \leq \pi/2$.

**Exercise 3.14.** Since $(-1)^n = e^{-in\pi}$, one has

$$\hat{g}(\omega) = \sum_n h[n]e^{in\omega} = \hat{h}(\omega + \pi).$$

If $h$ is a low-pass filter, then $g$ is a high-pass filter.

**Exercise 3.15.** If $g = f \ast h$ then $|g|_1 \leq \|f\|_1|h|_1$ and we can exchange the summations order as follow

$$\hat{g}(\omega) = \sum_n e^{-in\omega} \sum_p f[p]h[n-p] = \sum_p f[p]e^{-ip \omega} \sum_n e^{-i\omega(n-p)}h[n-p]$$

$$= \sum_p f[p]e^{-i \omega p} \sum_n e^{-i \omega n}h[n] = \hat{f}(\omega)\hat{h}(\omega)$$

where the second line is obtained by change of variable $n-p \to p$ in the summation.

**Exercise 3.16.** a) Since $\hat{h}(\omega)\hat{h}^{-1}(\omega) = \hat{\delta}(\omega) = 1$, one has $h^{-1} = 1/\hat{h}$.

b) Up to translation we suppose that $h$ is causal. The filter $h^{-1}$ and $h$ have finite support if and only if $\hat{h}(\omega) = P(e^{-i\omega})$ where $P(z)$ and $z^k/P(z)$ are polynomial for some $k \in \mathbb{N}$. This can only happens if $P(z)$ is a monomial $P(z) = az^p$, so that $h[n] = a\delta[n-p]$.

**Exercise 3.17.** a) One has

$$|h(\omega)|^2 = \prod_{k=1}^K \frac{|a_k^* - e^{-i\omega}|^2}{|1 + e^{-i\omega}|^2}$$

and one verifies that $|a_k^* - e^{-i\omega}|^2 = 1 + |a_k|^2 + 2\text{Re}(a_k e^{-i\omega}) = |1 + e^{-i\omega}|^2$.

b) $\{h[n-m]\}_m$ is an orthogonal basis if and only if for all $m, m'$

$$\langle h[n-m], h[n-m'] \rangle = h \ast h[m-m'] = \delta[n-m'].$$

Taking the Fourier transform of this relation leads to $|h(\omega)|^2 = 1$ for all $\omega$.

**Exercise 3.18.** a) For $h_0[n] = a^n1_{[0,\infty)}[n]$, one has the following geometrical sum

$$\hat{h}_0(\omega) = \sum_n (ae^{-i\omega})^n = \frac{1}{1-ae^{-i\omega}}.$$

Let $h_p(\omega) = (1-ae^{-i\omega})^{-p}$. Observe that

$$h'_p(\omega) = \frac{-pae^{-i\omega}}{(1-ae^{-i\omega})^{p+1}} = \sum_n h_p[n](-in)e^{-in\omega}.$$

The inverse Fourier transform of $h_{p+1}(\omega) = (1-ae^{-i\omega})^{-p-1}$ thus satisfies

$$h_{p+1}[n] = \frac{(n+1)h_p[n+1]}{ap}.$$
Iterating on this relation with \( h_1[n] = a^n 1_{[0, +\infty)}[n] \) gives

\[
h_p[n] = \frac{h_1[n + p - 1]}{a^p (p - 1)!} \prod_{k=1}^{p-1} (n + k)
\]

b) Taking the Fourier transform of the recursion formula shows that

\[
\hat{g}(\omega) = \hat{h}(\omega) \hat{f}(\omega) \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k e^{-i k \omega}}{\sum_{k=0}^{M} b_k e^{-i k \omega}}.
\]

c) Let \( a_k \) be the roots of the polynomial \( \sum_{k=0}^{M} b_k z^{-k} \) with multiplicity \( p_k \). Then

\[
\hat{h}(\omega) = P(e^{-i \omega}) \prod_{k} (1 - a_k e^{-i \omega})^{-p_k},
\]

where \( P \) is a polynomial. If one has \( |a_k| < 1 \) for all \( k \), then using question a), each \( (1 - a_k e^{-i \omega})^{-p_k} \) is the Fourier transform of a stable causal filter, and so is \( h \).

If there exists \( l \) such that \( |a_l| = 1 \), then it can be written \( a_l = e^{i \alpha} \) so \( \hat{h}(\alpha) = +\infty \) and the filter can therefore not be stable.

If, there exists \( l \) with \( |a_l| > 1 \) then let \( h_l(\omega) = (1 - a_l e^{-i \omega})^{-p_l} \). Observe that

\[
h_l(-\omega) = (1 - a_l e^{i \omega})^{-p_l} = a_l^{-p_l} e^{i \omega p_l} (a_l^{-1} e^{-i \omega} - 1)^{-p_l}.
\]

Its inverse Fourier transform is \( h_l[-n] \) and with question (a) we verify that it is not a causal filter. To prove that \( h \) is then not a causal filter, one can decompose

\[
\hat{h}(\omega) = \sum_{k=0}^{L} \frac{Q_k(e^{-i \omega})}{(1 - a_k e^{-i \omega})^{p_k}}
\]

where each \( Q_k \) is a polynomial of degree strictly smaller than \( n_k \) and observe that the component for \( k = l \) is not causal so \( h \) cannot be causal.

**Exercise 3.19.** One has, using the inverse Fourier transform,

\[
\hat{f}[2n] = \frac{1}{2N} \sum_{k=0}^{2N-1} \hat{f}[k] e^{\frac{2\pi i}{N} k 2n} = \frac{1}{N} \sum_{k=0}^{N/2-1} \hat{f}[k] e^{\frac{2\pi i}{N} k n} + \frac{1}{N} \sum_{k=3N/2+1}^{2N-1} \hat{f}[k] e^{\frac{2\pi i}{N} k n} + \frac{1}{N} \hat{f}[N/2] e^{\frac{2\pi i}{N} N/2n},
\]

and one thus has

\[
\hat{f}[2n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] e^{\frac{2\pi i}{N} k n} = \hat{f}[n].
\]

**Exercise 3.20.** a) One has

\[
\hat{x}(\omega) = \sum_n y[Mn] e^{-i \omega n} = \sum_n (y \cdot h)[n] e^{-i \omega n/M} = \frac{1}{2\pi} \hat{y} \ast \hat{h}(\omega/M)
\]

where we have use the convolution result of Exercise 3.7 and where

\[
h[n] = \sum_k \delta[n - k M]
\]
which satisfies, in the sense of distributions
\[ \hat{h}(\omega) = \frac{2\pi}{M} \sum_k \delta(\omega - 2\pi k/M). \]

This shows that
\[ \hat{x}(\omega) = \frac{1}{M} \int_{-\pi}^{\pi} y(\xi) \sum_k \delta(\omega/M - \xi - 2\pi k/M) d\xi = \frac{1}{M} \sum_{k=0}^{M-1} \hat{y}(\omega - 2k\pi/M). \]

b) If for \( \omega \in [-\pi, \pi], \hat{y}(\omega) = 0 \) for \( \omega \notin [-\pi/M, \pi/M] \), then
\[ \hat{y}(\omega) = \hat{x}(M\omega)\hat{\phi}_M(\omega) \quad \text{where} \quad \hat{\phi}_M(\omega) = M1_{[-\pi/M, \pi/M]}(\omega). \]

Let \((x \uparrow M)[Mn] = x[n]\) and \((x \uparrow M)[k] = 0\) if \( k \mod M \neq 0\) be the up-sampled signal. The Fourier transform of \( x \uparrow M \) is \( \hat{x}(M\omega) \), so that
\[ y = (x \uparrow M) \ast \phi_m \quad \text{where} \quad \phi_M[n] = \frac{\sin(n\pi/M)}{n\pi/M}. \]

Exercise 3.21. a) One has, using the change of variable \( r = n - pN \),
\[ \hat{f}_p[k] = \sum_{p \in \mathbb{Z}} \sum_{n=0}^{N-1} f_d[n - pN]e^{-\frac{2\pi}{N} kn} = \sum_{r \in \mathbb{Z}} f_d[r]e^{-\frac{2\pi}{N} r} = \hat{f}_d \left( \frac{2k\pi}{N} \right). \]

Using the sampling Theorem 3.1, one gets
\[ \hat{f}_p[k] = \hat{f}_d \left( \frac{2k\pi}{N} \right) = \frac{1}{s} \sum_{l} \hat{f} \left( \frac{2k\pi}{Ns} - \frac{2\ell\pi}{s} \right). \]

b) In order to have
\[ s\hat{f}_p[k] \approx \hat{f} \left( \frac{2k\pi}{Ns} \right), \]
one needs that \( \omega_0 \ll \pi/s \). To be able to interpolate without too much aliasing \( \hat{f} \) from the values \( \hat{f} \left( \frac{2k\pi}{N} \right) \), one needs that \( t_0 \ll Ns \). Since a function cannot be compactly supported in both time and space, no exact interpolation formula is possible.

c) The Fourier transform \( \hat{f}(\omega) \) is proportional to the convolution of the indicator of \([-\pi, \pi]\) convolved with itself 4 times. Its support is thus \([-4\pi, 4\pi]\).

Exercise 3.22. a) One has
\[ \hat{f}[\ell] = \frac{N}{2} \left( \delta(\ell - k) + \delta(\ell + k) \right). \]
So
\[ f_a[n] = e^{\frac{2\pi}{N} kn}. \]

b) One has \( g[n] = (f[n] + f^*[n])/2 \), and using a change of variable \( n \rightarrow -n \) in the summation gives
\[ \hat{g}[k] = \hat{f}[k]/2 + \sum_n f^*[n]e^{-\frac{2\pi}{N} kn} = \hat{f}[k]/2 + \hat{f}[-k]^*/2. \]
c) The definition of \( \hat{f}_a \) shows that \( g = \text{Re}(f_a) \) satisfies
\[
\hat{g}[k] = \left( \hat{f}_a[k] + \hat{f}_a[-k]^* \right)/2 = \hat{f}[k],
\]
and hence \( g = f \).

Exercise 3.23. One has
\[
\langle e_{k_1}[n_1] e_{k_2}[n_2], e_{k_1}'[n_1] e_{k_2}'[n_2] \rangle = \sum_{n_1,n_2} e_{k_1}[n_1] e_{k_2}[n_2] e_{k_1}'[n_1] e_{k_2}'[n_2] = \langle e_{k_1}, e_{k_1}' \rangle \langle e_{k_2}, e_{k_2}' \rangle = \delta[k_1 - k_1'] \delta[k_2 - k_2'].
\]

Exercise 3.24. We define the sub-images
\[
\forall 0 \leq k_1, k_2 < L, \quad \forall 0 \leq n_1, n_2 < M, \quad f_k[n] = f[n + kM].
\]
One has
\[
f[n] = \sum_k f_k[n - kM],
\]
so that
\[
f \ast h[n] = \sum_k f_k[n - kM] \ast h[n] = \sum_k (\hat{f}_k \circ \hat{h})[n - kM]
\]
where we have denoted by \( \hat{f}_k \) the image of size \((2M - 1) \times (2M - 1)\) obtained by zero-padding from \( f_k \). This allows one to compute \( f \ast h \) using \( L^2 \) circular FFT of size \((2M - 1)^2\), followed by \( 4N \) additions to reconstruct the full image. The overall complexity of this overlap add method is thus approximately \( 8KN \log(M) \).

The complexity of a direct evaluation of the convolution is \( K'NM^2 \), where \( K' \sim 2 \) (1 addition and 1 multiplication).

For \( K = 6 \) and \( K' = 2 \), using the overlap-add algorithm is better in the range of \( M \) such that
\[
48N \log_2(M) \leq 2NM^2 \iff M^2 / \log_2(M) \geq 24
\]
which we found numerically to be \( M \geq 9 \).

Exercise 3.25. The computation of \( \hat{f} \) is performed by applying a 1D FFT to each direction of the 3D array. This requires
\[
KN(\log(N_1) + \log(N_2) + \log(N_3)) = KN \log(N)
\]
operations.

3 Chapter 4

Exercise 4.1. a) One has
\[
Sf(u, \xi) = \int e^{i\phi(t)} g(t - u) e^{-i\xi t} dt = \hat{h}(\xi) \quad \text{where} \quad h(t) = e^{i\phi(t)} g(t - u).
\]
Using Parseval conservation of energy and the fact that $|g| = 1$, 
\[ \int |Sf(u, \xi)|^2 d\xi = 2\pi \int |h(t)|^2 dt = 2\pi \int |g(t)|^2 = 2\pi. \]

b) One has, using the fact that $\hat{h}'(\xi) = i\xi \hat{h}(\xi)$ and Parseval conservation of inner product, 
\[ \int \xi |Sf(u, \xi)|^2 d\xi = \int \xi \hat{h}(\xi) \hat{h}^*(\xi) d\xi = 2i\pi \int h'(t) h^*(t) dt, \]
and thus, expanding $h'(t) h^*(t)$, 
\[ \int \xi |Sf(u, \xi)|^2 d\xi = 2\pi \int \phi'(t) |g(t-u)|^2 dt + 2\pi \int g'(t-u) g^*(t-u) dt, \]
and the second integral vanishes because $g'$ is an odd function.

If one interpret $P_u(\xi) = |Sf(u, \xi)|^2/(2\pi)$ as a probability density then this result proves that the average frequency of this density for a fixed $t$ is equal to the averaged instantaneous frequency $\phi'(u)$ over a neighborhood of $t$ defined by the density $|g(t-u)|^2$.

**Exercise 4.2.** The formula (2.32) for the Fourier transform of the Gaussian implies 
\[ Ag(\tau, \gamma) = \frac{1}{\sqrt{\pi \sigma^2}} \int \exp \left( \frac{1}{2\sigma^2} \left[ (v + \tau/2)^2 + (v - \tau/2)^2 \right] \right) e^{-iv\gamma} dv \]
\[ = \frac{1}{\sqrt{\pi \sigma^2}} e^{-\tau^2/(4\sigma^2)} \int e^{-v^2} e^{-iv\gamma} dv = \exp \left( -\frac{\tau^2}{4\sigma^2} - \frac{\gamma^2 \sigma^2}{4} \right). \]

**Exercise 4.3.** The reconstruction (4.66) is exact if and only if for all signal $f$, 
\[ f = \log(a) \sum_{j=1}^{J} \frac{1}{a_j} f \otimes \psi_j^* \otimes \psi_j + \frac{1}{a^2 C_\psi} f \otimes \phi_j^* \otimes \phi_j. \]
Taking the finite discrete Fourier transform of this relation leads to 
\[ \forall k, \quad \hat{f}[k] = \hat{f}[k] \cdot \left( \log(a) \sum_{j=1}^{J} \frac{1}{a_j} |\psi_j[k]|^2 + \frac{1}{a^2 C_\psi} |\phi_j[k]|^2 \right), \]
and hence the result.

**Exercise 4.4.** One can write the discrete Fourier transform in (4.27) 
\[ S[f|m, l] = e^{-i2\pi ml/N} f \ast g_l[n] \text{ with } g_l[n] = g[-n] e^{-i2\pi ln/N}. \]
It is thus computed with $N$ convolutions. Since $g_l$ has a support of size $L$, the fast overlapp-add convolution algorithm of Section 3.3.4 computes each convolution with $O(N \log L)$ operations.

The windowed Fourier transform is thus computed with $O(N^2 \log L)$ operations.

**Exercise 4.5.** a) Applying (4.19) to $f = g_{u, \xi_0}$ proves 
\[ g_{u, \xi_0} = \frac{1}{2\pi} \int \langle g_{u, \xi_0}, g_{u, \xi} \rangle g_{u, \xi} d\xi. \]
Taking the inner product with $g_{v,\nu}$ proves

$$K(u_0, v, \xi_0, \nu) = \frac{1}{2\pi} \int K(u_0, u, \xi_0, \xi)K(u, v, \xi, \nu)d\xi,$$

which implies $PP\Phi = P\Phi$ so that $P$ is a projector on $\text{Im}(P)$, which is exactly, by Theorem 4.2 the functions $\Phi(u, \xi)$ that are windowed Fourier transform of some $f \in L^2$.

The projector $P$ is symmetric because

$$K(u_0, u, \xi_0, \xi) = K(u, u_0, \xi, \xi_0).$$

A projector $P$ is orthogonal (which means that $\text{Ker}(P) \perp \text{Im}(P)$) if and only it is a symmetric operator $P = P^*$ where $P^*$ is the dual (conjugated transposed) operator of $P$ for the inner product

$$\langle \Phi_1, \Phi_2 \rangle = \frac{1}{2\pi} \int \int \Phi_1(u, \xi)\Phi_2(u, \xi)^*d\xi.$$

Indeed for all $\Phi_1 \in \text{Ker}(P)$ and $\Phi_2 = P\Phi_2 \in \text{Im}(P)$,

$$\langle \Phi_1, \Phi_2 \rangle = \langle \Phi_1, P\Phi_2 \rangle = \langle P\Phi_1, \Phi_2 \rangle = 0.$$

b) Since an orthogonal projector $P$ is 1-Lipschitz, computing $P\tilde{S}f$ reduces the quantization error because

$$\|Sf - P\tilde{S}f\| = \|P(Sf - \tilde{S}f)\| \leq \|Sf - \tilde{S}f\|.$$

Exercise 4.6. One has

$$b(t) = \frac{1}{C_\psi} \int_0^{s_0} f * \psi\ast(t) \frac{ds}{s^2} + \frac{1}{C_\psi s_0} f * \phi\ast s_0.$$

Taking the Fourier transform leads to

$$\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C_\psi} \left[ \int_0^{s_0} |\hat{\psi}(s\omega)|^2 \frac{ds}{s^2} + \frac{1}{s_0} |\hat{\phi}s_0\ast|^2 \right].$$

Using the fact that $|\hat{\psi}(s\omega)|^2 = s|\hat{\psi}(s\omega)|^2$ and

$$|\hat{\phi}s_0\ast|^2 = \int_1^{+\infty} |\hat{\psi}(s_0s\omega)|^2 \frac{ds}{s} = \int_1^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{ds}{s}$$

leads to the result.

Exercise 4.7. Using Plancherel formula and then Fubini,

$$\|\phi\|^2 = \frac{1}{2\pi} \int |\hat{\phi}(\omega)|^2 d\omega = \frac{1}{2\pi} \int \int_1^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{ds}{s} d\omega

= \frac{1}{2\pi} \int_1^{+\infty} \left( \int |\hat{\psi}(s\omega)|^2 d\omega \right) \frac{ds}{s}. $$

The change of variable $s\omega \to s$ in the inner integral, and the fact that $\|\psi\| = 1$ leads to

$$\|\phi\|^2 = \int_1^{+\infty} \left( \frac{1}{2\pi} \int |\hat{\psi}(\omega)|^2 d\omega \right) \frac{ds}{s^2} = \int_1^{+\infty} \frac{ds}{s^2} = 1.$$
Exercise 4.8. Let 
\[ b(t) = \frac{1}{C} \int_0^{+\infty} f * \psi_s(t) \frac{ds}{s^{3/2}}. \]
Its Fourier transform reads, after a change of variable $\xi = s\omega$, 
\[ \hat{b}(\omega) = \frac{\hat{f}(\omega)}{C} \int_0^{+\infty} \sqrt{s} \hat{\psi}(s \omega) \frac{ds}{s^{3/2}} = \frac{\hat{f}(\omega)}{C} \int_0^{+\infty} \hat{\psi}(\xi) \frac{ds}{s} = \hat{f}(\omega). \]

Exercise 4.9. a) If $f$ is $C^p$, then $\hat{f}^{(p)}(\omega) = (i\omega)^p \hat{f}(\omega)$ and thus the inverse Fourier transform formula gives the result.

b) On the set $\Omega_\rho = \{ z \mid \text{Im}(z) > \rho \}$, one has 
\[ |(i\omega)^p \hat{f}(\omega)e^{iz\omega}| \leq |\omega|^p |\hat{f}(\omega)|e^{-\rho\omega}, \]
which is integrable. Using classical result of holomorphy under the sign $\int$, one sees that $f^{(p)}(z)$ is holomorphic on $\Omega_\rho$ for all $\rho > 0$.

c) One has 
\[ f^{(p)}(x + iy) = \frac{1}{2\pi} \int_0^{+\infty} (i\omega)^p \hat{f}(\omega)e^{ix\omega}e^{-\rho \omega} d\omega. \]
A wavelet transform of $f$ is written over the Fourier domain 
\[ Wf(u,s) = \frac{1}{2\pi} \int \hat{f}(\omega)\sqrt{s} \hat{\psi}(s \omega)e^{i\omega u} d\omega \]
so one sees that $f^{(p)}$ is indeed a Fourier transform if one sets 
\[ \hat{\psi}(\omega) = 1_{[0, +\infty]}(\omega)(i\omega)^p e^{-\omega}. \]
Using the inverse Fourier transform, and the fact that $\hat{h}^{(p)}(\omega) = (i\omega)^p \hat{h}(\omega)$, one has 
\[ \psi(t) = h^{(p)}(t) \quad \text{where} \quad h(t) = \frac{1}{2\pi} \int_0^{+\infty} e^{-\omega} e^{i\omega t} = \frac{1}{2\pi} \frac{1}{1 - it}, \]
so 
\[ \psi(t) = \frac{i \rho!}{2\pi (t + i)^p}. \]

Exercise 4.10. For $f(t) = \cos(\theta(t))$ with $\theta(t) = a \cos(bt)$, the width $s$ of the window is small enough if 
\[ s^2 |\theta''(t)| = s^2 ab^2 |\cos(bt)| \ll 1 \]
and thus $s^2 ab^2 \ll 1$.

For $\theta(t) = \cos(\theta_1(t)) + \cos(\theta_2(t))$ with $\theta_1(t) = a \cos(bt)$ and $\theta_2(t) = a \cos(bt) + ct$, there is enough frequency resolution if 
\[ |\theta_1'(t) - \theta_2'(t)| = |c| \geq \frac{\Delta_\omega}{s} \]
and enough spacial resolution if 
\[ s^2 |\theta_1''(t)| = s^2 |\theta_2''(t)| = s^2 ab^2 \ll 1. \]
This shows that one needs 
\[ \sqrt{\frac{c}{ab^2}} \ll \Delta_\omega \ll 1. \]
Exercise 4.15. One has,

\[ \int Pf(u, \xi)du = \frac{1}{\|f\|^2} \int |f(u)|^2du|\hat{f}(\omega)|^2 = |\hat{f}(\omega)|^2 \]

Using Plancherel conservation of energy formula, one has

\[ \frac{1}{2\pi} \int Pf(u, \xi)d\xi = |f(u)|^2 \frac{1}{\|f\|^2} \int |\hat{f}(\omega)|^2d\omega = |f(u)|^2. \]

Since \( Pf \) is not quadratic in \( f \) (it is of degree 4), one cannot apply Theorem 4.11.

Exercise 4.16. Example 4.20 shows that \( P_{\theta}f \) is a spectrogram with a Gaussian window \( g_{\mu} \) if \( \theta(u, \xi) = g_{\sigma}(u)g_{\sqrt{\sigma}(\xi)} \),

where we have used (4.125) for the last equality. If \( \sigma\beta = \frac{1}{2} \), then one can take \( \mu = \sqrt{\frac{2}{\pi}} \).

Otherwise, if \( \sigma\beta > \frac{1}{2} \), one decomposes \( \theta = \theta_0 \star \theta_1 \) where

\[ \theta_0(u, \xi) = g_{\mu/\sqrt{\sigma}(u)}g_{\sqrt{\sigma}(\xi)} \quad \text{and} \quad \theta_1(u, \xi) = g_{\sigma - \mu/\sqrt{\sigma}(u)}g_{\beta - \sqrt{\sigma}(\xi)} \]

where \( \mu \) is chosen so that \( \sigma - \mu/\sqrt{\sigma} > 0 \) and \( \beta - \sqrt{\sigma} > 0 \). This shows that \( P_{\theta} \) is a smoothing with \( \theta_1 \) of \( P_{\theta_0} \) which is a spectrogram and hence positive.

Exercise 4.17. Since \( \{g_n(t)\}_{n \in \mathbb{N}} \) is an orthonormal basis,

\[ \sum_{n=0}^{+\infty} g_n^*(t)g_n(t') = \delta(t-t'). \]

Indeed, for all \( f \in L^2 \)

\[ \int f(t) \sum_{n=0}^{+\infty} g_n^*(t)g_n(t')dt = \sum_{n=0}^{+\infty} \langle f, g_n \rangle g_n(t') = f(t') = \int f(t) \delta(t-t')dt. \]

It results that

\[ \sum_{n=0}^{+\infty} g_n^*(u - \tau/2)g_n(u + \tau/2) = \delta(\tau). \]

Since \( PVg_n(u, \xi) \) is the Fourier transform of \( g_n^*(u - \tau/2)g_n(u + \tau/2) \) with respect to \( \tau \), it results that

\[ \sum_{n=0}^{+\infty} PVg_n(u, \xi) = 1. \]

Exercise 4.18. One has

\[ A(t) = \int (\xi - \phi'(t))^2 PVf(t\xi)d\xi = \int h(\tau) \left( \int (\xi - \phi'(t))^2 e^{-i\xi\tau}d\xi \right) d\tau \]

where

\[ h(\tau) = a(t + \tau/2)a(t + \tau/2)e^{i(\phi(t+\tau/2) - \phi(t-\tau/2))}. \]

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In the sense of distribution, one has the following equality
\[ \int (\xi - \phi'(t))^2 e^{-i\xi t} \, d\xi = 2\pi [-\delta''(\tau) + 2i\phi'(t)\delta'(\tau) + \phi'(t)^2\delta(\tau)] \]
where \( \delta, \delta', \delta'' \) are the Dirac distribution and its first and second derivatives. This shows that
\[ A(t) = 2\pi [-h''(0) + 2i\phi'(t)h'(0) + \phi'(t)^2h(0)]. \]
After computing the derivatives of \( h \) and simplification, one finds
\[ A(t) = -\pi(a(t)a''(t) - a'(t)^2), \]
which is the result.

4 Chapter 5

Exercise 5.1. One has a union of \( K \) orthogonal bases
\[ \mathcal{D} = \{ \phi_p \}_{0 \leq p < KN} = \bigcup_{i=0}^{K-1} \{ \phi_{Kp+i} \}_{0 \leq p < N}, \]
so that \( \mathcal{D} \) is tight frame of frame bound \( KN \) since
\[ \sum_{p=0}^{KN-1} |\langle f, \phi_p \rangle|^2 = \sum_{i=0}^{K-1} N \left( \sum_{p=0}^{N-1} |\langle f, \phi_{Kp+i}/\sqrt{N} \rangle|^2 \right) = \sum_{i=0}^{K-1} N\|f\|^2 = KN\|f\|^2. \]

Exercise 5.2. A change of variable \( t \to Kt \) gives
\[ \langle f, \phi_p \rangle = \int_0^1 f(t)e^{2i\pi pt/K} \, dt = K \int_0^{1/K} f(Kt)e^{2i\pi pt} \, dt = \int_0^1 f_K(t)e_p(t) \, dt, \]
where \( \{e_p(t) = e^{-2i\pi pt}\}_p \) is an orthonormal basis of \( L^2[0, 1] \). Let us define \( f_K(t) = f(Kt) \) for \( t \in [0, 1/K] \) and \( f_K(t) = 0 \) otherwise. For \( K \geq 1 \), we get
\[ \sum_p |\langle f, \phi_p \rangle|^2 = \sum_p |\langle f_K, e_p \rangle|^2 = |f_K|^2 = K\|f\|^2. \]
So \( \{\phi_p\}_p \) is a tight frame of \( L^2[0, 1] \) of frame bound \( K \).

Exercise 5.3. The operator \( \Phi\Phi^* \) is bounded on \( H = \text{Im}(\Phi) \) which is of finite dimension, so \( B < +\infty \). If \( A = 0 \), then it exists \( f \neq 0 \) in \( H \) such that \( \sum_n |\langle f, \phi_n \rangle|^2 = 0 \) so that \( \langle f, \phi_n \rangle = 0 \) for all \( n \). Since \( \{\Phi_n\}_n \) is generator of \( H \), one has \( f = 0 \) which is a contradiction.

Exercise 5.4. One has
\[ \text{tr}(U_1U_2) = \sum_{i,j} U_1[i,j]U_2[j,i] = \text{tr}(U_2U_1). \]

Exercise 5.5. One has \( \Phi f[p] = f \odot h \) where \( h = \delta[\cdot] - \delta[\cdot - 1] \). As the discrete Fourier basis diagonalizes \( \Phi \), the frame bounds are
\[ A = \min_{\omega \neq 0} |\hat{h}[\omega]|^2 = \min_{\omega \neq 0} 4\sin^2 \left( \frac{\pi}{N} \omega \right) = 4\sin^2(\pi/N) \]
and $B = 4$ (if $N$ is even). One has $A/B → 0$ when $N → +∞$, so the frame is unstable when $N$ is large.

**Exercise 5.6.** If one does not constrain $∥φ_p∥$, one can choose $\{δ_0/N, Nδ_1, δ_2, …, δ_{N−1}\}$. If one constrains $∥φ_p∥ = 1$, one can choose $φ_p[n] = φ[n + p] \mod N$ so that $Φf = f * φ$ and thus one should impose

$$\sum_ω |φ[ω]|^2 = N \quad \text{and} \quad |φ[ω]| > 0.$$  

This can be achieved by setting $φ[ω] = 1/N$ for $ω ≠ 0$ and $φ[0] = \sqrt{N − (N − 1)/N^2}$. This implies

$$A = \min_ω |φ[ω]|^2 = 1/N^2 → 0$$  

and

$$B = \max_ω |φ[ω]|^2 = N − (N − 1)/N^2 → +∞.$$  

**Exercise 5.7.** If $x ∈ \text{Null}(U^*)$ and $y = Uz ∈ \text{Im}(U)$, then

$$(x, y) = (x, Uz) = (U^*x, z) = 0$$  

so that $\text{Null}(U^*) = \text{Im}(U)^⊥$.

**Exercise 5.8.** One has

$$Φ_mf[p] = ⟨f, φ_{m+p}⟩ = f ∗ φ_m[p] \quad \implies \quad Φ^*Φ = \sum_m Φ^*_mΦ_m$$  

where $Φ_m$ is a convolution operator, whose eigenvalues are $Φ_m[ω]$, and eigenvectors are the discrete Fourier vectors. The eigenvalues of $Φ^*Φ$ are thus $\sum_m |Φ_m[ω]|^2$.

**Exercise 5.9.** Let

$$φ_{k,p}(t) = g(t − 2pω/ω0)e^{ikω0t} \quad \text{so} \quad φ^*_{k,p}(ω) = ̂g(ω − kω0)e^{iω/ω02pπ}.$$  

Since $\{e^{iω/ω02pπ}\}_p$ is an orthonormal basis of each interval $[−ω0/2, ω0/2] + kω0$, $\{1/2πφ_{k,p}\}_k$ is an orthogonal basis of $L^2(\mathbb{R})$. Since $f → 1/2π̂f$ is an isometry, this prove that $\{φ_{k,p}\}_k$ is also an orthogonal basis.

**Exercise 5.10. a)** Since $\{1/\sqrt{K} e^{2πi/nk}\}_{0 ≤ k < K}$ is an orthogonal basis of the signal supported in $I = [−K/2, K/2 − 1] + mM$, and $g[n − mM]f[n]$ is supported in $I$, one has

$$\sum_n |g[n − mM][f[n]|^2 = \sum_{k=0}^{K−1} |⟨f, 1/√K g_{mn,k}⟩|^2.$$  

b) Summing over $m$ leads to

$$K \sum_n |f[n]|^2 \sum_{m=0}^{N/M−1} |g[n − mM]|^2 = \sum_{k,m} |⟨f, g_{m,k}⟩|^2,$$

and hence the result of Theorem 5.18.

**Exercise 5.11.** For $m = 2$, the filters are written, with the convention $z = e^{−iω}$

$$h(ω)/√2 = \cos^3(ω/2)e^{−iω/2} = P(z) = (z−1 + 3z + z^2)/8.$$  

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\[
\hat{g}(\omega)/\sqrt{2} = -i \sin(\omega/2)e^{-i\omega/2} = Q(z) = z/2 - 1/2.
\]
Applying Bezout algorithm to the polynomials \((1 + 3z + 3z^2 + z^3)/8\) and \((z^2 - z)/2\) leads to
\[
(8z - 7z^2) \times P(z) + \left(\frac{17}{4}z + 5z^2 + \frac{7}{4}z^3\right)Q(z) = 1
\]
and the reconstruction filters are given by
\[
\hat{h}(\omega)/\sqrt{2} = 8z - 7z^2 + 1/2 \quad \text{and} \quad \hat{g}(\omega) = \frac{17}{4}z - 5z^2 + \frac{7}{4}z^3.
\]

**Exercise 5.12.** For \(m = 3\), one has
\[
\hat{h}(\omega)/\sqrt{2} = \cos^4(\omega/2) \quad \text{and} \quad \hat{g}(\omega) = \frac{17}{4}z - 5z^2 + \frac{7}{4}z^3
\]
and one can choose
\[
\hat{g}(\omega)/\sqrt{2} = \sin^2(\omega/2) = Q(z) = (z - 2 + z^{-1})/4.
\]
If \(\hat{h} = h\), then
\[
\hat{g}(\omega) = \frac{2 - |\hat{h}(\omega)|^2}{\hat{g}^*(\omega)} = 2 - \frac{1 - \cos^4(\omega/2)}{\sqrt{2}\sin^2(\omega/2)} = \sqrt{2}[1 + \cos^2(\omega/2)] = \sqrt{2}Q(z)
\]
where \(Q(z) = 1 + (3 + 2 + z^{-1})/4\), so \(\hat{g}/\sqrt{2} = [1/4, 3/2, 1/4]\).

**Exercise 5.13.** Using the same proof as Theorem 5.18 and exercise 5.10, with \(M = 1\), \(K = N\), one sees that
\[
\left\{g_{m,\ell}[n] = g[n - m] \exp\left(\frac{2i\pi \ell n}{N}\right)\right\}_{m,\ell}
\]
is a tight frame of frame bound
\[
N \sum_n |g[n - m]|^2 = N\|g\|^2 = N.
\]
Theorem 5.5 implies that the dual frame is \(\{g_{m,\ell}/N\}_{m,\ell}\).

The reproducing kernel reads
\[
K(m, m', \ell, \ell') = \langle g_{m,\ell}, g_{m',\ell'} \rangle = \sum_n g[n - m]g[n - m'] \exp\left(\frac{2i\pi \ell n}{N}(\ell - \ell')\right).
\]

**Exercise 5.14.** As \(\beta \eta \to 2\pi\), the frame becomes less orthogonal \((A/B \to +\infty)\), so the dual vectors become more and more different from the primal ones, and hence the windows \(g\) and \(\tilde{g}\) differ more and more.

**Exercise 5.15. a)** One has
\[
\forall \pi \in [-\pi, \pi], \quad \hat{x}(\omega) = \frac{1}{s} \hat{f}(\omega/s)
\]
for \(s = T/K\), so \(\text{supp}(\hat{x}) \subset [-\pi/K, \pi/K]\).
b) Since \( x - \tilde{x} = W \)
\[
J(h) = E(\|x \ast h - x\|^2) = E(\|x \ast (h - \delta) + W \ast h\|^2).
\]
Since \( W \) is a white noise of variance \( \sigma^2 \) independant of \( x \)
\[
J(h) = E(\|x \ast (h - \delta)\|^2) + E(\|W \ast h\|^2).
\]
Since \( W \) is a white noise of variance \( \sigma^2 \) its power spectrum is \( \sigma^2 \). Let \( \hat{R}_x(\omega) \) be the power spectrum of the stationary random vector \( x[n] \),
\[
J(h) = \frac{1}{2\pi} \int_\pi^{-\pi} (\hat{R}_x(\omega))[1 - \hat{h}(\omega)]^2 + \sigma^2[\hat{h}(\omega)]^2 \, d\omega.
\]
To minimize \( J(h) \) for each \( \omega \) we minimize the value under the integral, which is obtained with
\[
\hat{h}(\omega) = \frac{\hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega)} \quad \text{and} \quad J(h) = \frac{1}{2\pi} \int_\pi^{-\pi} \frac{\sigma^2\hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega)} \, d\omega \leq \frac{\sigma^2}{K}.
\]

c) In this case \( \tilde{x} = x \ast h_p + W \) so
\[
J(h) = E(\|x \ast (h \ast h_p - \delta)\|^2) + E(\|W \ast h\|^2)
\]
\[
= \frac{1}{2\pi} \int_\pi^{-\pi} (\hat{R}_x(\omega))[1 - \hat{h}(\omega)\hat{h}_p(\omega)]^2 + \sigma^2[\hat{h}(\omega)]^2 \, d\omega.
\]
The minimum is obtained with
\[
\hat{h}(\omega) = \frac{\hat{R}_x(\omega)\hat{h}_p(\omega)}{\sigma^2 + \hat{R}_x(\omega)\hat{h}_p(\omega)^2} \quad \text{and} \quad J(h) = \frac{1}{2\pi} \int_\pi^{-\pi} \frac{\sigma^2\hat{R}_x(\omega)}{\sigma^2 + \hat{R}_x(\omega)\hat{h}_p(\omega)^2} \, d\omega.
\]
Since \( \hat{h}_p(\omega) = (1 - e^{-i\omega})^{-p} \),
\[
J(h) \leq \frac{\sigma^2}{2\pi} \int_{-\pi/K}^{\pi/K} |2\sin(\omega/2)|^{2p} \, d\omega \leq \frac{\sigma^2}{K} \frac{\pi^{2p}}{K^{2p} (2p + 1)}.
\]
For a fixed \( K \geq \pi \), increasing \( p \) decreases the error.

**Exercise 5.16.** a) Each \( \{\phi_s(t - ns - ks/K)\}_n \) is an orthogonal basis of \( U_s \), the set of signals with Fourier support included in \( [-\pi/s, \pi/s] \). This proves that \( \{\phi_s(n/Ks)\}_n \) is a tight frame with frame bound \( K \).

b) One has
\[
Pa[n] = \sum_{n'} a[n']H[n, n'] \quad \text{where} \quad H[n, n'] = \langle \phi_s(t - n/Ks), \frac{1}{K}\phi_s(t - n'/Ks) \rangle,
\]
and, using Plancherel formula,
\[
H[n, n'] = \frac{1}{2K} s^{1/2} 1_{[-\pi/s, \pi/s]}(\omega) e^{-in/Ks\omega}, \quad s^{1/2} 1_{[-\pi/s, \pi/s]}(\omega) e^{-in'/Ks\omega} = \frac{s}{2K} \int_{-\pi/s}^{\pi/s} e^{i(n - n')/Ks\omega} \, d\omega = h[n - n'] \quad \text{where} \quad h[n] = \frac{\sin(\pi n/K)}{\pi n}.
\]
c) Since \( \hat{h} = 1_{[-\pi/K, \pi/K]} \), one has
\[
\text{Im}(P) = \{a \setminus h \ast a = a\} = \{a \setminus \text{supp}(\hat{a}) \subset [-\pi/K, \pi/K]\}.
\]

d) Theorem 3.5 proves that \( s^{-1/2}f \ast \phi_s(ns) = f(ns) \).
e) Let \( a[n] = f \ast \phi_s(ns_0) \). For \( \omega \in [-\pi, \pi] \), one has
\[
\hat{a}(\omega) = \frac{1}{s_0} \mathcal{F}(f \ast \phi_s)(\omega/s_0) \implies \text{supp}(\hat{a}) \subset [-\pi/K, \pi/K]
\]
where \( \mathcal{F} \) is the Fourier transform. This shows that \( a \in \text{Im}(\Phi) \) and hence \( Pa = a \).

One has
\[
\]

One then has
\[
E(|W \ast h[Kn]|^2) = \sigma^2 \sum_k |h[k]|^2 = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} |\hat{h}(\omega)|^2 = \frac{\sigma^2}{K} \to 0 \quad \text{when} \quad K \to +\infty.
\]

**Exercise 5.17.**
a) Theorem 5.11 applied to the generator \( \frac{1}{s\sigma |\psi^{(k)}(u)|} = \phi_n(u) \) proves the result.
b) \( \psi \) has \( p \) vanishing moments if and only if \( \hat{\psi}^{(k)}(0) = 0 \) for all \( k < p \). Thus, since
\[
\sum_j |\hat{\psi}^{(k)}(2^j\omega)|^2 > A > 0,
\]
one has that \( \hat{\psi}^{(k)}(0) = 0 \) and thus \( \tilde{\psi} \) has also \( p \) vanishing moments.
c) One has for \( g = \{g_j(u)\}_j \)
\[
(P_V g)_j(u) = \sum_{j'} g_{j'}(u') (\psi_{2^j, u}, \hat{\psi}_{2^{j'}, \omega}) du' = \sum_{j'} g_{j'}(u') \cdot (\psi_{j} \ast \hat{\psi}_{j'})(u - u') du.
\]
This shows that
\[
(P_V g)_j = \psi_j \ast \sum_{j'} g_{j'} \ast \hat{\psi}_{j'}.
\]

**Exercise 5.18.** We prove the right hand side of the inequality
\[
\sum_{j=\alpha}^{\beta} |\hat{\psi}^{(2^j\omega)}|^2 = \frac{1}{2} \sum_{j=\alpha}^{\beta} |\hat{\psi}^{(2^{j-1}\omega)}| |\hat{\psi}^{(2^j-1\omega)}|^2
\leq B \sum_{j=\alpha}^{\beta} (1 - |\hat{\psi}^{(2^{j-1}\omega)}| |\hat{\psi}^{(2^j-1\omega)}|^2)
\leq B \sum_{j=\alpha}^{\beta} (|\hat{\psi}^{(2^{j-1}\omega)}|^2 - |\hat{\psi}^{(2^j\omega)}|^2) = B (|\hat{\psi}^{(2^{\alpha-1}\omega)}|^2 - |\hat{\psi}^{(2^\beta\omega)}|^2).
\]
Since \( \hat{\psi}(0) = 1 \) and \( \phi(\omega) \) tends to zero as \( \omega \) increases, letting \( \alpha \) and \( \beta \) go to \( -\infty \) and \( +\infty \) proves that \( \sum_j |\hat{\psi}^{(2^j\omega)}|^2 \leq B \). A similar proof applies to the left hand side.
Exercise 5.19. a) One has

\[ Zg_{n,k}(u, \xi) = \sum_{\ell} e^{2i\pi \xi \ell} 1_{[0,1]}(u - \ell - n)e^{2i\pi k(u - \ell)}. \]

Since \(1_{[0,1]}(u - \ell - n)\) is zero unless \(\ell = -n\), one obtains

\[ Zg_{n,k}(u, \xi) = e^{-2i\pi \xi n} e^{2i\pi uk}. \]

Since \(\{e^{-2i\pi \xi n} e^{2i\pi uk}\}_{n,k}\) is an orthogonal basis of \(L^2[0,1]^2\), the Zak transform transforms an orthogonal basis into an orthogonal basis, and is thus an unitary transforms.

b) One has

\[ \int_0^1 Zf(u, \xi) d\xi = \sum_{\ell} \int_0^1 e^{2i\pi \xi \ell} d\xi f(u - \ell) = f(u). \]

Note that this formula is valid to recover \(f(u)\) for \(u \in [0, 1]\), but it can be extended to arbitrary \(u \in \mathbb{R}\) using the fact that \(Zf(u + 1, \xi) = e^{2i\pi \ell} Zf(u, \xi)\).

c) Let \(g_{n,k}(t) = e^{2i\pi k} g(t - n)\). Similarly to question a), one proves that \(Zg_{n,k}(u, \xi) = e^{-2i\pi \xi n} e^{2i\pi uk}(Zg)(u, \xi)\).

One has, using the fact that the Zak transform is unitary, and then Plancherel formula for Fourier series of functions defined on \([0, 1]^2\),

\[ \sum_{n,k} |\langle f, g_{n,k} \rangle|^2 = \sum_{n,k} |\langle Zf, Zg_{n,k} \rangle|^2 = \sum_{n,k} \left| \int \int e^{-2i\pi \xi n} e^{2i\pi uk}(Zf)(u, \xi)(Zg)(u, \xi)dud\xi \right|^2 \]

\[ = \int \int |(Zf)(u, \xi)|^2 |(Zg)(u, \xi)|^2 dud\xi. \]

This shows that if \(\forall (u, \xi), \quad A \leq |(Zg)(u, \xi)|^2 \leq B\), then, using the fact that \(Z\) is an isometry,

\[ A\|f\|^2 = A\|Zf\|^2 \leq \sum_{n,k} |\langle f, g_{n,k} \rangle|^2 \leq B\|Zf\|^2 = B\|f\|^2, \]

which proves that \(\{g_{n,k}\}_{n,k}\) is a frame with bounds \(A\) and \(B\).

d) Denoting \(\Phi f[n, k] = \langle f, g_{n,k} \rangle\) the frame operator, we have proved that

\[ \|\Phi f\|^2 = \int \int |(Zf)(u, \xi)|^2 |(Zg)(u, \xi)|^2 dud\xi, \]

so that

\[ Z(\Phi \Phi^* f)(u, \xi) = |(Zg)(u, \xi)|^2 (Zf)(u, \xi) \]

and hence

\[ Z((\Phi \Phi^*)^{-1} f)(u, \xi) = \frac{(Zf)(u, \xi)}{|(Zg)(u, \xi)|^2}. \]

The dual frame \(\{\tilde{g}_{n,k}\}_{n,k}\) is defined as

\[ \tilde{g}_{n,k} = (\Phi \Phi^*)^{-1} g_{n,k} \]
and hence
\[ Z\tilde{g}_{n,k}(u,\xi) = \frac{(Zg_{n,k})(u,\xi)}{|(Zg)(u,\xi)|^2} = e^{-2i\pi\xi_n}e^{2i\pi uk} \]
\[ = \frac{e^{-2i\pi\xi_n}e^{2i\pi uk}}{(Zg)^*(u,\xi)} = (Zh_{n,k})(u,\xi) \]
where we have defined
\[ h_{n,k}(t) = e^{2i\pi k t}(t - n) \quad \text{where} \quad (Zh)(u,\xi) = (Zg)^*(u,\xi)^{-1}. \]

**Exercise 5.21.** We use the change of variables
\[ x = \rho \cos(\alpha) - u \sin(\alpha) \quad \text{and} \quad y = \rho \sin(\alpha) + u \cos(\alpha). \]
For \( k + \ell < p \), one can expand the monomial \( P(x,y) = x^k y^\ell \) as a polynomial in \( u \) of degree less than \( p \)
\[ P(x,y) = \sum_{t<p} A(\rho)u^t. \]
This shows that
\[ \langle P, \psi \rangle = \sum_{t<p} \int A(\rho) \left( \int u^t \psi(\rho \cos(\alpha) - u \sin(\alpha), \rho \sin(\alpha) + u \cos(\alpha)) \, du \right) \, d\rho = 0. \]

**Exercise 5.22.** a) One has
\[ \sum_k n_k^* n_k = \left( \begin{array}{cc} A & C \\ C & B \end{array} \right), \]
where, denoting \( \alpha_k = 2k\pi/K \),
\[ A = \sum_k \cos^2(\alpha_k), \quad B = \sum_k \sin^2(\alpha_k) \quad \text{and} \quad C = \sum_k \cos(\alpha_k) \sin(\alpha_k). \]
One has \( C = 0 \) and
\[ A = B = K/2 + \sum_k \cos(2\alpha_k) = K/2, \]
so \( \sum_k n_k^* n_k = K/2 \I_{d_2} \), which proves that \( \{n_k\}_k \) is a tight frame with frame bound \( K/2 \).

b) One has
\[ \psi^k = \langle [\partial_{\theta^x}, \partial_{\theta^y}], n_k \rangle \quad \Rightarrow \quad \hat{\psi}^k(\omega) = \hat{\theta}(\omega) \langle i[\omega_x, \omega_y], n_k \rangle. \]
This shows that
\[ \sum_{k=0}^K \sum_j |\hat{\psi}^k(2^j \omega)|^2 = \sum_j |\hat{\theta}(\omega)|^2 2^{2j} \sum_k |\langle [\omega_x, \omega_y], n_k \rangle|^2 \]
and hence the result since
\[ \sum_k |\langle [\omega_x, \omega_y], n_k \rangle|^2 = K/2 \langle \|\omega\|^2 \rangle. \]
Chapter 6

Exercise 6.1. a) It is an immediate consequence of Definition 6.1 of Lipschitz and uniform Lipschitz regularity.

b) For \( t \neq 0 \) \( f \) is \( C^1 \) so it is 1-Lipschitz, and \( |f(t)| \leq |t| \) so \( f \) is also 1-Lipschitz at 0.

If \( f \) is uniformly Lipschitz \( \alpha \) then there exists \( K > 0 \) such that for all \( (u, v) \in [-1, 1]^2 \),
\[ |f(u) - f(v)| \leq K |u - v|^{\alpha}. \]
For \( t_n = (n + 1/2)^{-1} \pi^{-1} \) we have \( f(t_n) = (-1)^n t_n \). So
\[ |f(t_n) - f(t_{n-1})| = t_n + t_{n-1} = \pi^{-1} \left( (n + 1/2)^{-1} + (n - 1/2)^{-1} \right) \sim 2\pi^{-1} n^{-1}. \]
Since \( t_n - t_{n-1} = \pi^{-1} (n + 1/2)^{-1} (n - 1/2)^{-1} \sim \pi^{-1} n^{-2} \) it results that \( |f(t_n) - f(t_{n-1})| \sim |t_n - t_{n-1}|^{1/2} \) and hence that \( \alpha \geq 1/2 \).

We now prove that \( f \) is indeed Lipschitz 1/2. If \( u \) and \( v \) have same sign and \( |1/u - 1/v| > 1 \) then there exists \( C > 0 \) with
\[ |f(u) - f(v)| \leq |f(t_n) - f(t_{n-1})| \leq C |t_n - t_{n-1}|^{1/2} \leq C |u - v|^{1/2}. \]
If \( u \) and \( v \) of same sign with \( |1/u - 1/v| \leq 1 \) and \( |u| > |v| \). It result that \( |u - v| \leq u^2 \), and
\[ |f(u) - f(v)| \leq |u - v| + |v| (|\sin u^{-1}| - |\sin v^{-1}|) \leq |u - v| + |v| (v^{-1} - u^{-1}). \]
Since \( |u - v| \leq 1 \)
\[ |f(u) - f(v)| \leq |u - v|^{1/2} + \frac{|u - v|}{|u|} \leq 2|u - v|^{1/2}. \]
If \( u \) and \( v \) have different signs, and \( |f(u)| \geq |f(v)| \), since \( |u - v| \leq 2 \) it results that
\[ |f(u) - f(v)| \leq 2|f(u)| \leq 2|u| \leq 2|u - v| \leq 2\sqrt{2} |u - v|^{1/2}. \]
It results that for any \((u, v) \in [-1, 1]^2\) there exists \( C' > 0 \) with
\[ |f(u) - f(v)| \leq C' |u - v|^{1/2} \]
and hence that \( f \) is uniformly Lipschitz 1/2 on \([-1, 1]\).

Exercise 6.2. a) The definition of \( f \) being \( \alpha \) Lipschitz can be written
\[ f(x) = f(u) + a(u)(x - u) + K(x, u)(x - u)^\alpha \quad (2) \]
with \( |K(x, u)| \) uniformly bounded in both \( x \) and \( u \), so
\[ |f(x) - f(u)|/(x - u) = a(u) + K(x, u)(x - u)^{\alpha-1} \]
which implies that \( f \) has derivatives at \( u \) with \( a(u) = f'(u) \), so (2) is re-written
\[ f(x) = f(u) + f'(u)(x - u) + K(x, u)(x - u)^\alpha \quad (3) \]
Then summing (2) for \( x = a + d, x = a \) and \( x = a - d \) shows that
\[ (f(a + d) - 2f(a) + f(a - d))/d = O(d^{\alpha-1}) \]
then applying (3) for \( x = a \) and both \( u = a - d \) and \( u = a + d \) shows that
\[ |f'(a - d) - f'(a + d)|/d = O(d^{\alpha-1}) \]
this shows that \( f' \) is Lipschitz \( \alpha - 1 \) and that this is uniform with the same bound.

b) If \( f(t) = t^2 \cos(1/t) \) then \( f'(t) = 2t \cos(1/t) - \sin(1/t) \). One has \( |f(t) - f(0)| \leq t^2 \) so that \( f \)
is 2-Lipschitz at 0, but \( \sin(1/t) \) is discontinuous at 0 of \( f' \) is not Lipschitz 1 at 0.

**Exercise 6.3.** One can take a hat function that is zero outsize \([-1, 1]\), that is linear over \([-1, 0]\)and \([0, 1]\), with \( f(0) = 1 \). It is 1-Lipschitz. We verify that \( |f(\omega)| = |\sin(\omega/2)|^2 |\omega|^{-2} \) so (6.4) is not satisfied.

**Exercise 6.4. a)** One has, for \( f(t) = \cos(\omega_0 t) \), and using Plancherel formula
\[
Wf(u, s) = \frac{1}{\sqrt{s}} \int \frac{1}{\sqrt{s}} \psi \left( \frac{t-u}{s} \right) f(t) \, dt \\
= \sqrt{s} \int \psi(s\omega) e^{-i\omega t} \left( \frac{1}{2} (\delta_{\omega_a}(\omega) + \delta_{-\omega_a}(\omega)) \right) d\omega \\
= \sqrt{s} \hat{\psi}(s\omega_0) \cos(u\omega_0),
\]
because \( \hat{\psi}(-\omega) = \hat{\psi}(\omega) \) because \( \psi(t) \) is symmetric.

b) One has,
\[
\frac{\partial Wf(u, s)}{\partial u} = 0 \iff u = u_k = \frac{k\pi}{\omega_0}.
\]
If \( \psi \) has \( p \) vanishing moments then \( |\hat{\psi}(\omega)| = O(\omega^p) \) so
\[
|Wf(u_k, s)| \leq \sqrt{s} |\hat{\psi}(s\omega_0)| = O(s^{p+1/2}).
\]

**Exercise 6.5.** For \( f(t) = |t|^\alpha \), one has, after the change of variable \( t \to t/s \),
\[
Wf(u, s) = \frac{1}{\sqrt{s}} \int |t|^{\alpha} \psi(t/s - u/s) dt = s^{\alpha+1/2} \int |t|^{\alpha} \psi(t - u/s) dt = Wf(u/s, 1).
\]
If \( \psi(t) \) is an antisymmetric wavelet, \( \psi(-t) = \psi(t) \), since \( f(t) \) is even the wavelet transform is antisymmetric and hence \( Wf(0, s) = 0 \) for all \( s \). In this case we thus can not derive the Lipschitz regularity at \( t = 0 \) from \( Wf(0, s) \).

**Exercise 6.6.** One has \( |f(t)| \leq |t|^\alpha \) so \( f \) is \( \alpha \)-Lipschitz. One has for \( t > 0 \)
\[
f'(t) = \alpha t^{\alpha-1} \sin(1/t^\beta) + \beta t^{\alpha-\beta-1} \cos(1/t^\beta),
\]
so \( f' \) is \( \alpha - \beta - 1 \)-Lipschitz.

The signal can be written \( f(t) = a(t) \cos(\theta(t)) \) with \( a(t) = |t|^\alpha \) and \( \theta(t) = |t|^{-\beta} \). The instantaneous frequency is \( \theta'(t) = \beta |t|^{-\beta-1} \). For such amplitudes and instantaneous frequencies one can apply the wavelet ridge calculations. Equation (4.109) shows that the large wavelet coefficients are along ridge curves \((u, s(u))\) defined by
\[
s(u) = \eta \theta'(u)^{-1} = \eta \beta^{-1} |u|^\beta+1
\]
where \( \eta \) is the center wavelet frequency.

Since \( f(t) \) is Lipschitz \( \alpha \) at 0, we derive from (6.20) in Theorem 6.4 that equation (6.21) is satisfied for \( \alpha' = \alpha \). One can not decrease \( \alpha' \) because at the ridge locations, (4.108) proves that
\[
|Wf(u, s)| = O(s^{1/2} a(u)) = O(s^{1/2} |u|^\alpha).
\]
Exercise 6.7. a) We proved in exercise 6.5 that if \( f(t) = |t|^\alpha \) then \( Wf(u, s) = s^{\alpha + 1/2}WF(u/s, 1) \). It results that the complex phase \( \Phi(u, s) \) of \( Wf(u, s) \) satisfies \( \Phi(u, s) = \Phi(u/s, 1) \). The lines of constant phase are thus along curves \((u, s)\) which satisfy \( u/s = \text{cst} \), and which converge to 0 when \( s \) goes to 0.

b) A modulus maxima point \( u_0 \) is an extremum of \( Wf(u, s) = f \ast \tilde{\psi}_s(u) \) with \( \tilde{\psi}_s(t) = s^{-1/2}\psi(-st) \) and thus satisfies \( \partial Wf(u_0, s)/\partial u = 0 \). But

\[
\frac{\partial Wf(u, s)}{\partial u} = -sf \ast \tilde{\psi}_s(u)
\]

with \( \psi^1(t) = \psi'(t) \). The modulus maxima computed with \( \psi \) are thus zeros of a wavelet transform computed with \( \psi' \). Let us now consider the analytic wavelet transform \( W^a f(u, s) = f \ast \tilde{\psi}_s(u) \) computed with the analytic complex wavelet

\[
\psi^a(t) = \psi'(t) + iH(\psi')(t)
\]

where \( H(\psi') \) is the Hilbert transform of \( \psi' \). The modulus maxima of \( W^a f(u, s) \) correspond to points \((u, s)\) where \( W^a f(u, s) \) has a real part equal to zero and hence a phase equal to \( \pi \). The modulus maxima points thus correspond to points of constant phase (equal to \( \pi \)) on this analytic complex wavelet transform.

Exercise 6.8. One has

\[
Wf(u, s) = \frac{1}{\sqrt{s}} \int_0^{+\infty} \psi \left( \frac{u - t}{s} \right) dt = \sqrt{s} \int_{-u/s}^\infty \psi(t) dt
\]

so that

\[
\frac{\partial Wf}{\partial u}(u, s) = 0 \iff \psi(-u/s) = 0.
\]

Since \( \psi \) is continuous and orthogonal to polynomials of degree less than \( p - 1 \), it has at least \( p \) zeros. Indeed, if this were not the case, there would exist less than \( p - 1 \) zero crossings \( \{t_k\}_k \) where \( \psi \) changes of sign at \( t_k \). Denoting \( P(t) = \prod_k(t-t_k) \) which is a degree less than \( p \), \( P\psi \) is non zero and of constant sign, so \( \langle f, P \rangle \neq 0 \), which is a contradiction.

Exercise 6.10. If \( f(t) = \int_{-t}^t d\mu_\infty(t) \) is a Cantor devil’s staircase then its singularities are located on the support of the Cantor positive measure \( d\mu_\infty \). Let \( t_0 \) be on the support of \( d\mu_\infty \). By construction, explained page 245, for any \( p > 0 \) there exists an integer \( q \) such that \( t_0 = q3^{-p} \), and either \( (q + 1)3^{-p} \) or \((q - 1)3^{-p} \) is also on the support of \( d\mu_\infty \). Suppose that it is the first case (the second one is treated similarly). The construction shows that

\[
\int_{(q-1)3^{-p}}^{q3^{-p}} d\mu_\infty(t) = 0, \quad \int_{q3^{-p}}^{(q+1)3^{-p}} d\mu_\infty(t) > 0, \quad \int_{(q+1)3^{-p}}^{(q+2)3^{-p}} d\mu_\infty(t) = 0. \tag{4}
\]

If \( \psi = -\theta' \) then according to (6.32)

\[
Wf(u, s) = s(f' \ast \tilde{\theta}_s)(u) = s \int s^{-1/2}\theta(s^{-1}(t-u)) d\mu_\infty(t). \tag{5}
\]

Let \( K \) be such that \( \theta \) has a support included in \([-K, K]\). To simplify explanations we suppose that \( \theta > 0 \) for \( |t| < K \). If \( s^{-1} = K3^{p+1} \) then \( \theta(s^{-1}(t-u)) \) has a support included in \([-3^{-p-1}, 3^{-p-1}] \).

It results from (4) and (5) that

\[
Wf((q - 1/2)3^{-p}, s_p) = 0, \quad Wf(q3^{-p}, s_p) > 0, \quad Wf((q + 3/2)3^{-p}, s_p) = 0.
\]

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So necessarily, \(|W f(u, s_p)|\) has at least one local maxima at position \(u_p\) in the interval \([q - 1/2)3^{-p}, (q + 3/2)3^{-p}\) and since \(t_0 = q3^{-p}\)

\[
\lim_{p \to +\infty} u_p = t_0.
\]

**Exercise 6.14.** a) One has \(\hat{\psi}(\omega) = |\omega|^2 \hat{\theta}(\omega)\) so that

\[
F(\omega) = \sum_{j=-\infty}^{+\infty} |\psi(2^j \omega)|^2 = \sum_{j=-\infty}^{+\infty} 2^{4j} e^{-2^{2j+1}|\omega|^2}.
\]

One has

\[
\sum_{j\leq 0} |\hat{\psi}(2^j \omega)|^2 \leq \sum_{j\leq 0} 2^{4j} = C < +\infty.
\]

Using the fact that \(2^{j+1} \leq 1 + (2j + 1) \log(2)\), one has

\[
\sum_{j>0} |\hat{\psi}(2^j \omega)|^2 \leq \sum_{j>0} e^{4j \log(2) - 1 + (2j + 1) \log(2)} |\omega|^2 \leq C' \sum_{j<0} e^{4j \log(2) - 2|\omega|^2 \log(2)}.
\]

This shows that for \(0 \leq |\omega|^2 \leq \alpha < 2\), \(F(\omega) \leq B\) and also

\[
F(\omega) \geq \sum_{j} 2^{4j} e^{-2^{2j+1}\alpha} = A > 0.
\]

This proves that for \(0 \leq |\omega|^2 \leq \alpha\), \(A \leq F(\omega) \leq B\). Using the fact that \(F(2\omega) = F(\omega)\), this bound is extended to all \(\omega \in \mathbb{R}^2\).

b) For \(\psi = \Delta \theta\), where \(\Delta\) is the Laplacian, one has

\[
Wf(u, 2^j) = 0 \iff \Delta(f \ast \theta_j) = 0,
\]

so this is a multiscale crossing of the Laplacian edge detector.

Suppose that the image can locally be approximated by a straight-edge profile

\[
f(x_1, x_2) = \rho(x_1 \cos \alpha - x_2 \sin \alpha)
\]

(6)

where \(\rho(t)\) is a monotonous function with an inflection point at \(t = 0\). A direct computation shows that if \(\psi\) is the Laplacian of a Gaussian \(\theta\) then \(Wf(u, s) = 0\) when \(u\) is an inflection point of

\[
f \ast \theta_s(x) = \rho_s(x_1 \cos \alpha - x_2 \sin \alpha)
\]

which is located along a straight edge curve of angle \(\alpha\), \(x_1 \cos \alpha - x_2 \sin \alpha = t_0\) with \(\rho_s''(t_0) = 0\).

It thus corresponds to the position of modulus maxima point computed with the two partial derivative wavelets \(\psi^1\) and \(\psi^2\).

Suppose that \(f\) has a curved edge, which can be modeled with an angle \(\alpha\) which has a slow variation as a function of \((x_1, x_2)\). Then the location of the zero-crossings of \(Wf(u, s)\) are not identical to the modulus maxima of the first derivative wavelets because of second order terms that are not identical in both wavelet transforms. The zero-crossings are not exactly located at the inflection points of \(f \ast \theta_s(x)\) but close to these points if the curvature is small.
Exercise 6.15. One has, denoting \( \psi_s(t) = \psi(t/s) / \sqrt{s} \),
\[
E(WB(u_1, s)WB(u_2, s)) = E([B_h \ast \hat{\psi}_s(u_1)][B_h \ast \hat{\psi}_s(u_2)])
\]
\[
= \int \int E(B_h(t)B_h(t'))\hat{\psi}_s(u_1 - t)\hat{\psi}_s(u_2 - t')dt dt'
\]
\[
= \sigma^2 \int \int (|t|^{2H} + |t'|^{2H} - |t - t'|^{2H})\hat{\psi}_s(u_1 - t)\hat{\psi}_s(u_2 - t')dt dt'
\]
\[
= -\sigma^2 \int \int |t - t'|^{2H}\hat{\psi}_s(u_1 - t)\hat{\psi}_s(u_2 - t')dt dt'
\]
where we have use the fact that \( \int \psi_s = 0 \) to derive the last equality. Performing the successive changes of variables \( t - t' \rightarrow t \) and \( u_2 - t' \rightarrow t' \), one gets
\[
E(WB(u_1, s)WB(u_2, s)) = -\sigma^2 \int |t|^{2H} \hat{\psi}_s * \psi_s(u_1 - u_2 - t)dt
\]
which gives the result after the change of variable \( t'/s \rightarrow t' \) in the convolution.

6 Chapter 7

Exercise 7.1. a) One has
\[
\hat{\phi}(\omega) = \prod_{j \geq 0} \frac{\hat{h}(\omega/2^{j+1})}{h(\omega/2^{j+1})}.
\]
For each \( j \geq 0 \) and \( \ell \in \mathbb{Z} \), \( h(\omega/2^{j+1}) \) has a zero of order \( p \) at location \( 2\pi(2^j(2\ell + 1)) \). Remarking that any \( k \in \mathbb{Z}^+ \) can be written \( k = 2^j(2\ell + 1) \) with \( j \geq 0 \) and \( \ell \in \mathbb{Z} \), one sees that \( \hat{\phi}(\omega) \) has a zero of order \( p \) at each \( \omega = 2k\pi, k \neq 0 \).

b) One has, using Theorem 3.1
\[
A(\omega) = \sum_n n^q \phi(n)e^{in\omega} = \frac{1}{i^q} \frac{d^q}{d\omega^q} \left( \sum_n \phi(n)e^{in\omega} \right)
\]
\[
= \frac{1}{i^q} \frac{d^q}{d\omega^q} \left( \sum_n \hat{\phi}(\omega + 2n\pi) \right) = \frac{1}{i^q} \sum_n \hat{\phi}^{(q)}(\omega + 2n\pi)
\]
One thus have, for \( \omega = 0 \),
\[
\sum n^q \phi(n) = A(0) = \frac{1}{i^q} \sum_n \hat{\phi}^{(q)}(2n\pi) = \frac{1}{i^q} \hat{\phi}^{(q)}(0) = \int t^q \phi(t)dt,
\]
since \( \hat{\phi}^{(q)}(2n\pi) = 0 \) for \( n \neq 0 \).

Exercise 7.2. Let \( A(t) = \sum_n \phi(t - n) \). Its Fourier transform satisfies, using Theorem 2.4, in the sense of distributions
\[
\hat{A}(\omega) = \hat{\phi}(\omega) \sum_n e^{-in\omega} = \hat{\phi}(\omega)2\pi \sum_n \delta(\omega - 2k\pi) = 2\pi \sum_n \phi(2k\pi)\delta(\omega - 2k\pi).
\]
Using Exercise 7.1, one has that \( \hat{\phi}(2k\pi) = 0 \) for \( k \neq 0 \), and \( \hat{\phi}(0) = 1 \) so that \( \hat{A}(\omega) = 2\pi \delta(\omega) \) and hence \( A(t) = 1 \).
Exercise 7.3. We choose \(m\) to be even so that \(\hat{\phi}_m\) is a symmetric function

\[
\hat{\phi}_m(\omega)^{-2} = 1 + \sum_{k\neq 0} \frac{1}{(1 + 2k\pi/\omega)^{2m+2}}.
\]

For \(0 < \omega < a < \pi\), one has \(|2n\pi/\omega| \geq |n|(1 + \varepsilon)\) with \(0 < \varepsilon < 1\) so that

\[
|\hat{\phi}_m(\omega)^{-2}| \leq 1 + 2 \sum_{n \geq 1} \frac{1}{(n(2 + \varepsilon) - 1)^{2m+2}}.
\]

We split the sum into \(1 \leq n \leq 1/\varepsilon\) and \(1/\varepsilon < n\). For the first part, the sum is finite and one has

\[
\frac{1}{n(2 + \varepsilon) - 1)^{2m+2}} \to 0 \quad \text{when} \quad m \to +\infty.
\]

For the second part, one has

\[
\sum_{n > 1/\varepsilon} \frac{1}{(n(2 + \varepsilon) - 1)^{2m+2}} \leq \sum_{n > 1/\varepsilon} \frac{1}{(2n)^{2m+2}} \leq \int_{1/\varepsilon}^{+\infty} \frac{1}{(2x)^{2m+2}} \, dx \to 0
\]

when \(m \to +\infty\). This shows that \(\hat{\phi}_m(\omega) \to 1\) for \(m \in (-\pi, \pi)\). A similar derivation shows \(\hat{\phi}_m(\omega) \to 0\) for \(|\omega| > \pi\), so that \(\hat{\phi}_m \to \hat{\phi}\) almost everywhere. Using dominated convergence and Plancherel formula, this shows that \(\|\phi_m - \phi\| \to 0\).

Exercise 7.4. a) If \(K = 2\), then \(\text{Supp}(\phi) \subset [0, 1]\) and

\[
\phi(t) = m[0]\phi(2t) + m[1]\phi(2t - 1),
\]

so that

\[
\phi(0, \varepsilon_0 \varepsilon_1 \cdots \varepsilon_i) = m[0]\phi(0, \varepsilon_0, \varepsilon_1 \cdots \varepsilon_i) + m[1]\phi((\varepsilon_0 - 1) \cdot \varepsilon_1 \cdots \varepsilon_i).
\]

If \(\varepsilon_0 = 1\), then \(\phi(\varepsilon_0, \varepsilon_1 \cdots \varepsilon_i) = 0\) and if \(\varepsilon_0 = 0\) then \(\phi((\varepsilon_0 - 1) \cdot \varepsilon_1 \cdots \varepsilon_i) = 0\) so that one has

\[
\phi(0, \varepsilon_0 \varepsilon_1 \cdots \varepsilon_i) = m[\varepsilon_0]\phi(0, \varepsilon_1 \cdots \varepsilon_i)
\]

and by recursion one obtains

\[
\phi(0, \varepsilon_0 \varepsilon_1 \cdots \varepsilon_i) = m[\varepsilon_0] \cdots m[\varepsilon_i] \phi(0).
\]

b) If \(t = 0, \varepsilon_0 \varepsilon_1 \cdots\) is a dyadic number, then there exists some \(K\) such that \(\varepsilon_k = 0\) for \(k \geq K\). For \(m[0] > 1\), if \(\phi(0) \neq 0\), this implies that \(\phi(t)\) is infinite.

c) Exercise 7.5. a) The recursion formula is written over the Fourier domain as

\[
\hat{\phi}_{k+1}(\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega/2)\hat{\phi}_k(\omega/2)
\]

so that, using Plancherel formula

\[
ak_{k+1}[n] = \frac{1}{4\pi} \int |\hat{h}(\xi/2)|^2 |\hat{\phi}_k(\xi/2)|^2 e^{i n \xi} \, d\xi.
\]
Using the Poisson summation formula, Theorem 2.4, one has the following equality, in the sense of distributions
\[
\hat{a}_{k+1}(\omega) = \frac{1}{4\pi} \int |\hat{h}(\xi/2)|^2 |\hat{\phi}_k(\xi/2)|^2 \left( \sum_n e^{-in(\omega-\xi)} \right) d\xi \\
= \frac{1}{2} \sum_n |\hat{h}(\omega/2 - n\pi)|^2 |\hat{\phi}_k(\omega/2 - n\pi)|^2
\]

b) If \( \phi_k \to \phi \), then \( a_k \to a \) where \( a[n] = \langle \phi(t), \phi(t-n) \rangle \). If \( |\hat{h}(\omega)| \) is bounded, then \( P \) is a bounded linear operator on \( L^2(\mathbb{R}) \). Hence \( Pa_k \to Pa \) and thus \( Pa = a \), which means that \( a \) is an eigenvector of \( P \) for the eigenvalue \( 1 \).

One has
\[
\hat{\phi}_k(\omega) = \prod_{p=1}^k 2^{-1/2} \hat{h}(2^{-p}\omega) \hat{\phi}_0(\omega/2^k)
\]

Since \( \hat{\phi}_0(\omega/2^k) \to 1 \) when \( k \to +\infty \) and \( \hat{\phi}_k \) converges to \( \hat{\phi} \), one has
\[
\hat{\phi}(\omega) = \prod_{p \geq 1} 2^{-1/2} \hat{h}(2^{-p}\omega).
\]

**Exercise 7.6. a)** Since \( f \in V_L \), one has
\[
f(x) = \sum_k \langle f, \phi_L,k \rangle \phi_L,k
\]
and
\[
b[n] = \sum_k a_L[k] 2^{-L/2} \phi(k-n)
\]
so that \( b = 2^{-L/2} a_L \ast \phi_d \) where \( \phi_d[n] = \phi(n) \).

**b)** Using Theorem 3.1, one has
\[
\hat{\phi}_d(\omega) = \sum_k \hat{\phi}(\omega + 2k\pi).
\]
The filter \( \hat{\phi}_d^{-1} \) whose Fourier transform is \( 1/\hat{\phi}_d(\omega) \) is stable if \( \hat{\phi}_d(\omega) \geq A > 0 \).

**d)** If \( \{\hat{\phi}_{L,n}\}_n \) is an interpolation basis of \( V_L \) for the points \( 2^L n \), then
\[
f = \sum_n b[n] \tilde{\phi}_{L,n} = \sum_n a_L[n] \phi_{L,n}.
\]
The change from \( b \) to \( a_L \) is stable if and only if the basis \( \{\hat{\phi}_{L,n}\}_n \) is stable.

**Exercise 7.7. a)** A computation similar to the proof of Theorem 7.11 shows that the reconstruction property is equivalent to having
\[
\hat{h}(\omega + \pi)\hat{h}(\omega) + \hat{g}(\omega + \pi)\hat{g}(\omega + \pi) = 0
\]
and
\[
\hat{h}(\omega)\hat{h}(\omega) + \hat{g}(\omega)\hat{g}(\omega) = 2e^{-i\omega}.
\]
One verifies that for a given \( h \), the proposed filters satisfy the first condition, and that the second condition is equivalent to
\[
\hat{h}^2(\omega) - \hat{h}^2(\omega + \pi) = 2e^{-i\omega}.
\]

b) Denoting \( z = e^{-i\omega} \), this last condition is re-written as
\[
\left( \sum_k h[k]z^k \right)^2 - \left( \sum_k (-1)^kh[k]z^k \right)^2 = 2z^l.
\]
One verifies that in the left hand side of this equation, all terms of even degrees cancel out, so that \( \ell \) is necessarily odd.

c) For the Haar filter, \( \sqrt{2}\hat{h}(\omega) = 1 + e^{-i\omega} \), so that
\[
\hat{h}^2(\omega) - \hat{h}^2(\omega + \pi) = (1 + z)^2 - (1 - z)^2/2 = 2z
\]
which corresponds to the quadrature mirror filter condition with \( \ell = 1 \).

Exercise 7.8. A Daubechies wavelet \( \psi_{j,n} \) with \( p \) vanishing moments has a support of size 
\( 2^j(2p - 1) \). One thus concentrates on the scales \( 2^j \) such that \( 2^j(2p - 1) < \min_k |\tau_k - \tau_{k-1}| \) such that each wavelet intersects only one discontinuity. Since the wavelets are translated by \( 2^j \) with support of size \( 2^j(2p - 1) \), each singularity generates exactly \( 2p - 1 \) non zero coefficients. One should thus choose \( p = q + 1 \). If \( p < q + 1 \), then there are \( 2^j \) coefficients at each scale, that can be all non zero.

Exercise 7.9. a) Denoting \( u = 1_{(0,+\infty)} \) and \( v = 1_{(1, +\infty)} \), one has \( \theta_1 = u - v \). Denoting \( f^{(k)} \) the convolution \( k \) times of \( f \) with itself, one has
\[
\theta_m = (u - v)^{(m+1)} = \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} u^{(k)} v^{(m-k)}.
\]
One has \( u^{(2)} = [t]_+, u^{(3)} = ([t]_+)^2/2 \), and more generally, by induction, one proves that
\[
u^{(k)}(t) = \frac{([t]_+)^{k-1}}{(k-1)!} \]
and
\[
v^{(k)}(t) = \frac{([t-k]_+)^{k-1}}{(k-1)!}.
\]
One then verifies that
\[
u^{(i)} \ast v^{(j)}(t) = \frac{1}{(i+j+1)!} ([t-j]_+)^{i+j+1},
\]
so that
\[
\theta_m(t) = \frac{1}{m} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} ([t-j]_+)^{m}.
\]
b) One has, using the Fourier convolution theorem
\[
|\theta_m(\omega)| = |\theta_0(\omega)|^{m+1} = |2\sin(\omega/2)/\omega|^{m+1}
\]
One has using (7.9)
\[
B_m = \max_{\omega} \sum_k |\hat{\theta}_m(\omega + 2k\pi)|^2
= \max_{\omega} 2^m |\sin(\omega/2)|^{2m+2} \sum_k \frac{1}{|\omega + 2k\pi|^{2m+2}}.
\]
By comparison with an integral, one obtains
\[ B_m \geq |\sin(\omega/2)| \sum_{k \geq 0} \frac{1}{(\omega/(2\pi) + k)\pi} \geq |\sin(\omega/2)| \int_{\omega/(2\pi)}^{+\infty} \frac{dx}{(\pi x)^{2m+2}} \]
\[ = |\sin(\omega/2)| \left( \frac{2m + 1}{\omega(2\pi)} \right) \pi \rightarrow +\infty. \]

when \( m \to +\infty \) as long as \( 2 < \omega < 2\pi \).

**Exercise 7.10.** This proof is quite technical, and we detail a proof inspired by “Unimodular Wavelets for \( L^2 \) and the Hardy Space of \( H^2 \)” by Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee, and Jin Keun Seo.

If \( \{\psi_{j,n}\}_{j,n} \) is an orthogonal basis of \( L^2(\mathbb{R}) \), then for all \( f \in L^2(\mathbb{R}) \), using Parseval formula
\[ \hat{f} = \frac{1}{2\pi} \sum_{j,n} (\hat{f}, \hat{\psi}_{j,n}) \hat{\psi}_{j,n}. \]

This corresponds to, for all \( \xi \),
\[ \hat{f}(\xi) = \frac{2j}{2\pi} \sum_{j,n} \int \hat{f}(\omega) \hat{\psi}^*(2^j\omega) \psi(2^j\xi)e^{-ik2^j(\xi-\omega)} d\omega. \]

One has, using Poisson formula, Theorem 2.4, in the sense of distribution with respect to the variable \( \omega \), the equality
\[ \sum_n e^{-ik2^j(\xi-\omega)} = \frac{2\pi}{2^j} \sum_k \delta(\omega - \xi + 2^{-j}2k\pi), \]

and thus
\[ \hat{f}(\xi) = \sum_{j,n} \hat{f}(\xi - 2^{-j}2k\pi) \hat{\psi}^*(2^j\xi - 2k\pi) \hat{\psi}(2^j\xi) \]

and hence
\[ \hat{f}(\xi)(1 - \theta(\xi)) = \sum_{j,k \neq 0} \hat{f}(\xi - 2^{-j}2k\pi) \hat{\psi}^*(2^j\xi - 2k\pi) \hat{\psi}(2^j\xi) \]

where
\[ \theta(\xi) = \sum_j |\hat{\psi}(2^j\xi)|^2. \]

For \( \xi_0 > 0 \) (the same derivation applies for \( \xi_0 < 0 \) and for some \( \varepsilon > 0 \) small enough, we use \( \hat{f} = 1_I \) with \( I = [\xi_0 - \varepsilon, \xi_0 + \varepsilon] \), and integrate with respect to \( \xi \) to obtain
\[ \int_I |\theta - 1| \leq \sum_{|k2^{-j}| < \varepsilon} \left( \int_I |\hat{\psi}(2^j\xi)||\hat{\psi}(2^j\xi - 2k\pi)| d\xi, \right. \]

where we have used the fact that \( \hat{f}(\xi - 2^{-j}2k\pi) \neq 0 \) on \( I \) only if \( |k2^{-j}| < \varepsilon \). Using Cauchy-Schwartz, and a change of variable \( 2^j\xi \to \xi \) one obtains
\[ \int_I |\theta - 1| \leq 2^{-j} \sum_{2^{-j} < \varepsilon} A_j(\xi) \left( \sum_{|k| < 2^{-j}} B_{j,k}(\xi) \right), \]

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where

\[ A_j(\xi)^2 = \int_{2^j I} |\hat{\psi}(\xi)|^2 d\xi \quad \text{and} \quad B_{j,k}(\xi)^2 = \int_{2^j I} |\hat{\psi}(\xi + 2k\pi)|^2 d\xi. \]

We note that, if \(|k2^{-j}| < \varepsilon\), then all the intervals \(2^j I + 2k\pi\) are included in \(2^j [\xi_0 - 3\varepsilon \pi, \xi_0 + 3\varepsilon \pi]\), so that

\[ \sum_{|k| < \varepsilon 2^j} B_{j,k}(\xi)^2 \leq 2\varepsilon 2^j \int_{2^j I} |\hat{\psi}(\xi)|^2 d\xi. \]

If follows that

\[ \int_I |\theta - 1| \leq 2\varepsilon \sum_{2^{-j} \leq \varepsilon} \int_{2^j I} |\hat{\psi}(\xi)|^2 d\xi. \]

If \(\varepsilon\) is small enough with respect to \(\xi_0\), \(\varepsilon < \xi_0 / (12\pi)\), then all the intervals \(2^j [\xi_0 - 3\varepsilon \pi, \xi_0 + 3\varepsilon \pi]\) are disjoint and thus

\[ \frac{1}{|I|} \int_I |\theta - 1| \leq \frac{1}{\pi} \int_{(\xi_0 - 3\varepsilon \pi) / \varepsilon}^{\infty} |\hat{\psi}(\xi)|^2 d\xi. \]

If \(\hat{\psi}\) is continuous, then the left part of this inequality tends to \(\theta(\xi_0) - 1\) whereas the right part tends to 0 when \(\varepsilon \to 0\). This shows that \(\theta(\xi_0) = 0\) for all \(\xi_0 \neq 0\). When \(\hat{\psi} \in L^2(\mathbb{R})\), this results holds only for almost all points \(\xi_0 \neq 0\) that are Lebesgue points of \(\hat{\theta}\).

To show that the condition

\[ \sum_j |\hat{\psi}(2^j \omega)|^2 = 1 \quad (7) \]

is not sufficient, let us note that the requirement that \(\{\psi(t - n)\}_n\) is an orthogonal system is written

\[ \sum_k |\hat{\psi}(\omega + 2k\pi)|^2 = 1. \quad (8) \]

We note that if \(\hat{\psi}(\omega)\) satisfies (7) and (8), then \(\hat{\psi}(\omega/2)\) still satisfies (7) but not (8) anymore, take for instance

\[ \hat{\psi}(\omega) = 1_{[-2\pi,-\pi] \cup [\pi,2\pi]}. \]

**Exercise 7.11.** If \(\hat{\psi} = 1_I\), is an indicator function, \(\psi_{j,k}\) is an orthogonal basis if it satisfies

\[ \sum_j |\hat{\psi}(2^j \omega)|^2 = 1 \quad \text{and} \quad \sum_k |\hat{\psi}(\xi + 2k\pi)|^2 = 1. \]

This means that one has the disjoint unions

\[ \mathbb{R} = \bigcup_j 2^j I = \bigcup_k (I + 2k\pi). \]

One verifies that this is the case for the set

\[ I/\omega = [-32, -28] \cup [-7, -4] \cup [4, 7] \cup [28, 32]. \]

where in the following we denote \(\omega = \pi/7\). The base approximation space is

\[ V_0 = \text{Span}(\psi_{j,n})_{j<0,n \in \mathbb{Z}}. \]

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A function \( \phi \in V_0 \) thus satisfies
\[
\operatorname{Supp}(\hat{\phi}) \subset \bigcup_{j < 0} (2^j I) \subset J
\]
where
\[
J/\omega = [-16, -14] \cup [-8, -7] \cup [-4, 4] \cup [7, 8] \cup [14, 16].
\]
If \( \phi \in V_0 \), then the function
\[
\theta(\omega) = \sum_k |\hat{\phi}(\omega + 2k\pi)|^2,
\]
is supported in \( \bigcup_k (J + 2k\pi) \), and one verifies that
\[
\left( \bigcup_k (J + 2k\pi) \right) \cap [4\omega, 6\omega] = \emptyset
\]
which implies that \( \theta(\omega) = 0 \) on \( [4\omega, 6\omega] \). This means that \( \{\phi(t - n)\}_n \) cannot be an orthogonal basis of \( V_0 \) because otherwise \( \theta(\omega) = 1 \) for all \( \omega \).

**Exercise 7.12.** The Coiflet condition is written over the Fourier domain as
\[
\forall 0 < k < p, \quad \frac{d^k \phi(\omega)}{d\omega^k}(0) = 0.
\]
One has
\[
\sqrt{2}\hat{\phi}(2\omega) = \hat{h}(\omega)\hat{\phi}(\omega),
\]
and taking the \( k \)th derivative of this expression at \( \omega = 0 \) leads to the equivalent condition that
\[
\forall 0 < k < p, \quad \frac{d^k h(\omega)}{d\omega^k}(0) = 0,
\]
which corresponds to the following conditions on the coefficients of \( h \)
\[
\sum_n n^k h[n] = 0.
\]

**Exercise 7.13.** Using the same derivation as in Exercise 7.12, the discrete signal \( \psi_j \) has \( p \) vanishing moments if and only if
\[
\forall 0 \leq k < p, \quad \frac{d^k \psi_j(0)}{d\omega^k} = 0.
\]
Taking \( k \) derivative of
\[
\hat{\psi}_j(\omega) = \hat{g}(2^{j-L-1}\omega)H(\omega) \quad \text{where} \quad H(\omega) = \prod_{p=0}^{j-L-2} \hat{h}(2^p\omega),
\]
shows that this is indeed the case if
\[
\forall 0 \leq k < p, \quad \frac{d^k \hat{g}(0)}{d\omega^k} = \frac{d^k \hat{h}(\pi)}{d\omega^k} = 0
\]
which is equivalent to the continuous wavelet \( \psi \) having \( p \) vanishing moments, see Theorem 7.4.
Exercise 7.14. a) Since $\psi$ has $p \geq 1$ vanishing moments, one can decompose
\[ \hat{h}(\omega) = (e^{i\omega} + 1)P(e^{-i\omega})/2 \]
where $P$ is a polynomial (see (7.91)), so that
\[ \hat{h}_1(\omega) = 2\hat{h}(\omega)/(e^{i\omega} + 1) = P(e^{i\omega}) \]
is a polynomial. It is obvious that $2(e^{i\omega} - 1)\hat{g}(\omega)$ is a polynomial in $e^{i\omega}$.

b) One has
\[ \hat{\psi}(\omega) = \frac{\hat{g}(\omega/2)}{\sqrt{2}} \prod_{p \geq 2} \frac{\hat{h}(\omega/2^p)}{\sqrt{2}}. \]
One has the following equality
\[ i\omega = (e^{i\omega} - 1) \prod_{p \geq 1} \frac{1 + e^{i\omega/2^p}}{2}, \]
and since the Fourier transform of $\psi'$ satisfies $\hat{\psi}'(\omega) = i\omega\hat{\psi}(\omega)$, one can write
\[ \hat{\psi}'(\omega) = \frac{\hat{g}_1(\omega/2)}{\sqrt{2}} \prod_{p \geq 2} \frac{\hat{h}_1(\omega/2^p)}{\sqrt{2}}, \]
with
\[ \hat{h}_1(\omega) = \frac{2\hat{h}(\omega)}{e^{i\omega} + 1} \quad \text{and} \quad \hat{g}_1(\omega) = 2(e^{i\omega} - 1)\hat{g}(\omega). \]

c) The derivative coefficients are obtained by replacing $(h, g)$ by $(h_1, g_1)$ in the pyramid algorithm.

Exercise 7.15. a) Denoting $\hat{\psi}_a = \text{Re}(\psi_a)$, and using the result of Exercise 2.4, one has
\[ 2\hat{\psi}_a(\omega) = \hat{\psi}_a(\omega) + \hat{\psi}_a(-\omega) = \hat{\psi}_a(\omega)|\hat{h}(\omega/4 - \pi/2)|^2 + \hat{\psi}_a(-\omega)|\hat{h}(\omega/4 - \pi/2)|^2 \]
Since $\psi$ is a real wavelet, $\hat{\psi}(-\omega)^* = \hat{\psi}(\omega)$ and since $h$ is also real, $|\hat{h}(-\omega)| = |\hat{h}(\omega)|$, which leads to
\[ 2\hat{\psi}_a(\omega) = \hat{\psi}_a(\omega) \left(|\hat{h}(\omega/4 - \pi/2)|^2 + |\hat{h}(\omega/4 + \pi/2)|^2\right) = 2\hat{\psi}(\omega) \]
where we have used the fact that $|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2$ because $h$ is a quadrature filter. This proves that $\text{Re}(\psi_a) = \psi$.

c) Denoting $g_1$ as the finite filter such that $\hat{g}_1(\omega) = \hat{h}(\omega)|\hat{h}(\omega - \pi/2)|^2$, one has to switch from the ordinary computation of the detail coefficients
\[ d_{j+1} = (a_j \ast \tilde{g}) \downarrow 2 \]
where $a \downarrow 2$ is the sub-sampling by 2 operator, to the following computation
\[ d_{j+1} = \left((a_{j-1} \ast \tilde{g}_1) \downarrow 2 \ast \tilde{g}\right) \downarrow 2. \]

d) One can compute the coefficients $a_j[n] = \langle f, \phi_j,n_1(x_1)\phi_j,n_2(x_2) \rangle$ from $a_{j-1}$ by applying filtering by $\hat{h}[n_1]\hat{h}[n_2]$ and subsampling along each direction. Then the detail coefficients $d_j[n] = \langle f, \psi_j,n \rangle$ are computed by filtering/sub-sampling $a_{j-2}$ two times by $\hat{h}$ along the $x_2$ direction and...
then by filtering/sub-sampling by \( \tilde{g}_1 \) and then by \( \tilde{g} \). Similar computations allow one to obtain the other details coefficients \( d_k^k[n] = \langle f, \psi^k_{j,n} \rangle \) for \( k = 2, 3, 4 \).

e) We denote by
\[
\psi^{[1]}(x) = \psi(x_1)\phi(x_1), \quad \psi^{[2]}(x) = \phi(x_1)\psi(x_2) \quad \text{and} \quad \psi^{[3]}(x) = \psi(x_1)\psi(x_2)
\]
the classical 2D wavelets. One has for a real signal \( f \),
\[
\langle f, \psi^{[1]}_{j,n} \rangle = \text{Re} \langle \langle f, \psi^1_{j,n} \rangle \rangle, \quad \langle f, \psi^{[2]}_{j,n} \rangle = \text{Re} \langle \langle f, \psi^2_{j,n} \rangle \rangle,
\]
and
\[
\langle f, \psi^{[3]}_{j,n} \rangle = \text{Re} \langle \langle f, \psi^3_{j,n} \rangle \rangle + \text{Re} \langle \langle f, \psi^4_{j,n} \rangle \rangle.
\]
This implies that
\[
|\langle f, \psi^{[1]}_{j,n} \rangle|^2 \geq |\langle f, \psi^1_{j,n} \rangle|^2, \quad |\langle f, \psi^{[2]}_{j,n} \rangle|^2 \geq |\langle f, \psi^2_{j,n} \rangle|^2,
\]
and
\[
|\langle f, \psi^{[3]}_{j,n} \rangle|^2 + |\langle f, \psi^{[4]}_{j,n} \rangle|^2 \geq \frac{1}{2} (|\langle f, \psi^3_{j,n} \rangle| + |\langle f, \psi^4_{j,n} \rangle|) = \frac{1}{2} |\langle f, \psi^{[3]}_{j,n} \rangle|^2.
\]
This shows that
\[
\sum_{j,n,k} |\langle f, \psi^k_{j,n} \rangle|^2 \geq \frac{1}{2} \sum_{j,n,k} |\langle f, \psi^k_{j,n} \rangle| = \|f\|^2 / 2,
\]
and hence the frame is redundant.

The reverse inequality is more technical, see [108], Appendix A, for a proof. It uses the fact that if a function \( \hat{\psi} \) satisfies \( |\hat{\psi}(\omega)| = O((1 + |\omega|)^a) \) with \( s > 1/2 \) and \( |\hat{\psi}(\omega)| = O(|\omega|^a) \) with \( \alpha > 0 \), then \( \{\psi_{j,n}\}_{j,n} \) is a stable frame of its span. It is easy to show that the almost analytic wavelets \( \tilde{\psi}^k(\omega) \) satisfy these decay conditions.

**Exercise 7.16.** a) One notes that \( \phi_1 = 1_{[0,1]} \) satisfies the scaling equation
\[
\phi_1(t) = \phi_1(2t) + \phi_1(2t - 1),
\]
whereas \( \phi_1(t) = 1_{[0,1]}(t)(2t - 1) \) satisfies
\[
\phi_2(t) = \frac{1}{2} (\phi_2(2t) + \phi_2(2t - 1) - \phi_1(2t) - \phi_1(2t - 1)).
\]
One has \( \langle \phi_1, \phi_2 \rangle = 0 \) so that \( \{\phi_1(t-n), \phi_2(t-n)\}_n \) is an orthogonal basis of the function that are affine on each interval \([n, n+1]\).

b) One needs to apply the Gram-Schmidt orthogonalization process to
\[
\{\phi_1, \phi_2, t^2 1_{[0,1]}(t), t^3 1_{[0,1]}(t)\}
\]
to obtain
\[
\{\phi_1, \phi_2, \psi_1, \psi_2\}.
\]
They correspond to the Legendre polynomials on \([0,1]\), and
\[
\{\phi_1(t-n), \phi_2(t-n), \psi_1(t-n), \psi_2(t-n)\}_{n \in \mathbb{Z}}
\]
is an orthogonal basis of the functions that are polynomials of degree 3 on each interval \([n, n+1]\). Since they are orthogonal to \( \phi_1, \phi_2 \), they are orthogonal to polynomial of degree 1 on their support, and hence they have two vanishing moments (and \( \psi_2 \) has 3 vanishing moments).
Exercise 7.17. a) One has

\[ \int_0^1 \alpha^{\text{repl}}(t)\beta^{\text{repl}}(t) \, dt = \sum_k \int_0^1 \alpha(t - 2k)\beta^{\text{repl}}(t) \, dt + \sum_k \int_0^1 \alpha(t - 2k)\beta^{\text{repl}}(t) \, dt \]

\[ = \sum_k \int_{-2k}^{-2k+1} \alpha(t)\beta^{\text{repl}}(t) \, dt + \sum_k \int_{2k-1}^{2k} \alpha(t)\beta^{\text{repl}}(t) \, dt \]

\[ = \int_{-\infty}^{+\infty} \alpha(t)\beta^{\text{repl}}(t) \, dt \]

\[ = \sum_k \langle \alpha(t), \beta(t - 2k) \rangle + \sum_k \langle \alpha(t), \beta(2k - t) \rangle = 0. \]

b) We treat the case where \( \phi, \tilde{\phi} \) are symmetric about 1/2 and \( \psi, \tilde{\psi} \) are anti-symmetric about 1/2, which implies \( \psi(1 - t) = -\psi(t) \). Since the folded signals are 2-periodic, one only needs to consider, at a scale 2^j with \( j < 0 \), indexes \( 0 \leq n < 2^{-j+1} \). One thus has, if \( (j, n) \neq (j', n') \),

\[ \langle \psi_{j,n}(t), \tilde{\psi}_{j',n'}(t - 2k) \rangle = \langle \psi_{j,n}, \tilde{\psi}_{j',n'+21-j-k} \rangle = 0 \]

\[ \langle \psi_{j,n}(t), \tilde{\psi}_{j',n'}(2k - t) \rangle = -\langle \psi_{j,n}, \tilde{\psi}_{j',n'+(1-2k)2-j'} \rangle = 0 \]

which implies, using question a), that

\[ \langle \psi_{j,n}^{\text{repl}}, \psi_{j',n'}^{\text{repl}} \rangle = 0, \]

and similarly with the other inner products with scaling functions.

For a given function \( f \in L^2([0,1]) \), we extend it by zero outside \([0,1]\) and decompose it in the wavelet basis of \( L^2(\mathbb{R}) \)

\[ f = \sum_{j \in J_n} \langle f, \psi_{j,n} \rangle \tilde{\psi}_{j,n} + \sum_n \langle f, \phi_{j,n} \rangle \tilde{\psi}_{j,n}, \]

so that for \( t \in [0,1] \),

\[ f(t) = f^{\text{repl}}(t) = \sum_{j \in J_n} \langle f^{\text{repl}}, \psi_{j,n}^{\text{repl}} \rangle \tilde{\psi}_{j,n}^{\text{repl}} + \sum_n \langle f^{\text{repl}}, \phi_{j,n}^{\text{repl}} \rangle \tilde{\psi}_{j,n}^{\text{repl}}. \]

Exercise 7.18. a) In the following, we denote \( z = e^{-i\omega} \), and denote

\[ \hat{P}(\omega) = P(z) = \frac{R(z)R(z^{-1})}{Q(z)Q(z^{-1})}. \]

The perfect reconstruction property reads \( P(z) + P(-z) = 2 \), so that \( P(z) = \sum_{k \in \mathbb{Z}} p_k z^k \) necessarily satisfies \( p_{2k} = 0 \) for \( k \neq 0 \). Hence one can decompose

\[ P(z) = 1 + z \frac{C(z^2)}{D(z^2)} = \frac{C(z^2) + zD(z^2)}{D(z^2)} = \frac{R(z)R(z^{-1})}{Q(z)Q(z^{-1})} \]

where \( C \) and \( D \) are polynomials with no root in common. Thus by identification \( 2D(z^2) = R(z)R(z^{-1}) + R(-z)R(-z^{-1}) \), which corresponds to

\[ \hat{P}(\omega) = \frac{2|\hat{P}(\omega)|^2}{|\hat{P}(\omega)|^2 + |\hat{P}(\omega + \pi)|^2} \]

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where \( r(\omega) = R(z) \).

b) The constraint on \( r \) is rewritten \( R(z^{-1}) = z^{k-1}R(z) \), so that
\[
R(z)R(-z^{-1}) = -R(-z)R(z^{-1})
\]
and hence
\[
|\hat{r}(\omega) + \hat{r}(\omega + \pi)|^2 = (R(z) + R(-z))(R(z^{-1}) + R(-z^{-1}))
= R(z)R(z^{-1}) + R(-z)R(-z^{-1}) = |\hat{r}(\omega)|^2 + |\hat{r}(\omega + \pi)|^2
\]
which implies the factorization
\[
\hat{p}(\omega) = \frac{2|\hat{r}(\omega)|^2}{|\hat{r}(\omega) + \hat{r}(\omega + \pi)|^2}
\]
c) One can choose
\[
\hat{h}(\omega) = \sqrt{\frac{2}{5}} \frac{(1 + z)^5}{(1 + z)^5 + (1 - z)^5}.
\]
One verifies that
\[
(1 + z)^5 + (1 - z)^5 = 2 + 20z^2 + 10z^4
\]
whose roots are \( \pm i \sqrt{1 \pm \sqrt{5}}/\sqrt{5} \).

**Exercise 7.19. a)** Let \( h_{\text{new}} \) and \( \tilde{h}_{\text{new}} \) be defined as
\[
h_{\text{new}}[n] = (h[n] + h[n-1])/2 \quad \text{and} \quad (\tilde{h}_{\text{new}}[n] + \tilde{h}_{\text{new}}[n-1])/2 = \tilde{h}[n]
\]
so that
\[
\hat{h}_{\text{new}}(\omega) = \hat{h}(\omega) \frac{1 + e^{i\omega}}{2} \quad \text{and} \quad \tilde{h}_{\text{new}}(\omega) = \hat{h}(\omega) \frac{2}{1 + e^{i\omega}}.
\]
Since \( \hat{h}_{\text{new}}(\omega)\hat{h}_{\text{new}}(\omega) = \hat{h}(\omega)\hat{h}(\omega) \), the biorthogonality is conserved.

If \( h \) and \( \hat{h} \) have \( p \) and \( \hat{p} \) vanishing moments, then
\[
\hat{h}(\omega) = (e^{i\omega} + 1)^p P(e^{i\omega}) \quad \text{and} \quad \hat{h}(\omega) = (e^{i\omega} + 1)^{\hat{p}} \hat{P}(e^{i\omega})
\]
where \( P \) and \( \hat{P} \) are polynomials. This shows that \( h_{\text{new}} \) has \( p + 1 \) vanishing moments, whereas \( \tilde{h}_{\text{new}} \) has \( \hat{p} - 1 \) vanishing moments.

b) One has the following matrix expression for the balancing operations
\[
\begin{pmatrix}
\hat{h}_{\text{new}}(\omega) \\
\tilde{h}_{\text{new}}(\omega)
\end{pmatrix} =
\begin{pmatrix}
\hat{a}(\omega) & 0 \\
0 & 1/\hat{a}(\omega)
\end{pmatrix}
\begin{pmatrix}
\hat{h}(\omega) \\
\tilde{h}(\omega)
\end{pmatrix}
\]
where \( \hat{a}(\omega) = (1 + e^{i\omega})/2 \).

Denoting \( A = \hat{a}(\omega) \), and using the following decomposition of a scaling matrix
\[
\begin{pmatrix}
A & 0 \\
0 & 1/A
\end{pmatrix} =
\begin{pmatrix}
1 & A - A^2 \\
-1/A & 1
\end{pmatrix}
\begin{pmatrix}
1 & A - 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix},
\]
one sees that the balancing operation can be implemented with 4 lifting steps.

c) We start by the original filters
\[
h = [1] \quad \text{and} \quad \tilde{h} = [-1, 0, 9, 16, 9, 0, -1]/16,
\]
after one step of balancing, one obtains
\[ h_{\text{new}}^1 = [1, 1]/2 \quad \text{and} \quad \hat{h}_{\text{new}}^1 = [-1, 1, 8, 8, 1, -1]/8, \]

after one other step of balancing, one obtains
\[ h_{\text{new}}^2 = [-1, 2, 6, 2, -1]/4 \quad \text{and} \quad \hat{h}_{\text{new}}^2 = [1, 2, 1]/4, \]

which corresponds to the 5/3 biorthogonal wavelets.

**Exercise 7.21.** After a simple factorization, applying an odd-length filter of size \( K \) such that \( h[n] = h[K - n - 1] \) requires \( K - 1 \) additions and \( (K - 1)/2 \) multiplications.

The direct application 5/3 filters requires 6 additions and 3 multiplications (times \( N \) for a signal of length \( N \)), whereas the lifting implementation requires 4 additions and 4 multiplications.

The direct application 9/7 filters requires 14 additions and 7 multiplications (times \( N \) for a signal of length \( N \)), whereas the lifting implementation requires 8 additions and 4 multiplication.

**Exercise 7.22.** For an interpolation wavelet, one has \( \psi(x) = \phi(2x - 1) \) and hence \( \hat{h}(\omega) = e^{-i\omega}/2 \). One verifies that \( \delta = \hat{\phi} \) and
\[
\tilde{g}(\omega) = 2\tilde{g}(\omega + \pi)e^{-i\omega}
\]
satisfy the biorthogonality relations, so that
\[
\tilde{\psi}(t) = \sum_{k \in \mathbb{Z}} (-1)^k h[k]\delta \left( t - \frac{k + 1}{2} \right).
\]

For interpolation of order 4, one has
\[
h = [-1, 0, 9, 16, 9, 0, -1]/16.
\]

One verifies that
\[
\forall q < 4, \quad \sum_k (-1)^k h[k]k^q = 0
\]
so that the dual wavelet has 4 vanishing moments.

**Exercise 7.23.** Denoting \( \phi_0 \) the Daubechies orthogonal wavelet with \( p \) vanishing moments, one has \( \phi(\omega) = |\hat{\phi}_0(\omega)|^2 \). Following Y. Meyer in “Wavelets with Compact Support”, one has the following formula
\[
|\hat{\phi}_0(\omega)|^2 = 1 - \frac{(2p - 1)!}{[p - 1]!(2p - 1)!} \int_0^\infty \sin^{2p-1}(t)dt
\]
which is an even trigonometric polynomial of order \( 2p - 1 \). This can be explicitly computed as
\[
|\hat{\phi}_0(\omega)|^2 = \frac{1}{2} + \frac{1}{2} \left( \frac{(2p - 1)!}{(p - 1)!(4p - 1)!} \right)^2 \sum_{k=1}^p \frac{(-1)^{k-1} \cos((2k - 1)\omega)}{(p - k)!(p + k - 1)(2k - 1)}
\]
by expanding \( \sin^{2p-1}(t) \). When \( p \to +\infty \), one has for \( |\omega| \neq \pi/2 \),
\[
|\hat{\phi}_0(\omega)|^2 \to \frac{1}{2} + \frac{2}{\pi} \sum_{k \geq 1} \frac{(-1)^{k-1}}{2k - 1} \cos((2k - 1)\omega),
\]
which is the Fourier series expansion of \( 1_{[-\pi/2, \pi/2]}(\omega) \) for \( \omega \in [-\pi, \pi] \). This thus shows that
\[
|\hat{\phi}_0(\omega)|^2 \to 1_{[-\pi/2, \pi/2]}(\omega),
\]

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and hence \( \phi(t) \) converges in \( L^2(\mathbb{R}) \) toward \( \sin(\pi t)/(\pi t) \).

**Exercise 7.24.** We follow the proof of Theorem 7.22. For each \( t = 2^j n + h \) where \( |h| \leq 2^j \), we write
\[
|f(t) - P_{V_j} f(t)| \leq |f(t) - f(t-h)| + |P_{V_j} f(t) - P_{V_j} f(t-h)|.
\]
Since \( f \) is Lipshitz \( \alpha \) regular,
\[
|f(t) - f(t-h)| \leq C_f |h|^\alpha \leq C_f 2^{\alpha j}
\]
where \( C_f \) is the Lipshitz constant. One also has
\[
P_{V_j} f(t) - P_{V_j} f(t-h) = \sum_{n=\infty}^{\infty} \left( f(2^j(n+1)) - f(2^j n) \right) \theta_{j,h}(t-n)
\]
where
\[
\theta_{j,h} = \sum_{k=1}^{\infty} (\Phi_j(t-h-2^j k) - \Phi_j(t-2^j k)).
\]
As in the proof of the proof of Theorem 7.22, since \( \phi \) has exponential decay,
\[
\sum_{n=\infty}^{\infty} |\theta_{j,h}(t-n)| \leq C_\phi
\]
and hence
\[
|P_{V_j} f(t) - P_{V_j} f(t-h)| \leq C_\phi \max_n |f(2^j(n+1)) - f(2^j n)| \leq C_\phi C_f 2^{\alpha j}.
\]

**Exercise 7.25.** Let \( \phi(x) \) be a 1D interpolation function. We define the 1D transforms \( \tau_0(x) = 0 \) and \( \tau_1(x) = 2x - 1 \). For each set \( \{\varepsilon_k\}_{k=1}^p \) of \( p \) binary values \( \varepsilon_k \in \{0,1\} \), we define
\[
\psi^\varepsilon(x_1, \ldots, x_p) = \prod_{k=1}^{p} \phi(\tau_{\varepsilon_k}(x_k)),
\]
where we denote \( 0 \leq \varepsilon < 2^p \) the number whose binary expansion is \( \{\varepsilon_k\}_{k=1}^p \). Then \( \psi^0 \) is an interpolating scaling function, and \( \{\psi^\varepsilon\}_{0\leq\varepsilon<2^p} \) defines \( 2^p - 1 \) interpolating wavelets that can be used to analyze continuous functions defined on \( \mathbb{R}^p \).

**Exercise 7.28.** We do the derivation here in 1D. The foveated signal \( T f(x) \) is obtained from \( f(x) \) by a spatially varying convolution with a filter scaled by \( t \) around position \( t \), \( g(\cdot/t)/t \), where \( g \) is a symmetric smooth low pass function
\[
T f(x) = \int_{-\infty}^{+\infty} f(t)g(x/t - 1) \frac{dt}{t}.
\]
The operator is written over the wavelet domain as
\[
T f = \sum_{j,j',n,n'} K_{j,j',n,n'}(f, \psi_{j,n}) \psi_{j',n'}
\]
where
\[
K_{j,j',n,n'} = \langle T \psi_{j,n}, \psi_{j',n'} \rangle = \iint \psi_{j,n}(t) \psi_{j',n'}(x) g(x/t - 1) \frac{dt}{t} dx.
\]
It is shown in “Wavelet Foveation” by Chang, Mallat and Yap that if \( g \) and \( \psi \) have compact support, with \( \psi \) regular with vanishing moments, then \( K_{j, j', n, n'} \) decay fast when \( |j - j'| \) or \( |n - n'| \) increases. Together with the fact that \( K_{j, j', n, n'} \) depends only on \( j - j' \), this shows that one can approximate the operator as a diagonal one

\[
Tf = \sum_{j, n} \lambda_n(f, \psi_{j,n})\psi_{j,n}
\]

One can show that if \( g \) is of class \( C^\alpha \), then \( \lambda_n = O(|n|^{-(\alpha + 1)}) \).

7 Chapter 8

Exercise 8.1. For \( A = 2^{j-L} \), one has that \( \{\theta_j[m - nA]\}_n \) is orthonormal if and only if

\[
\langle \theta_j[m], \theta_j[m - nA] \rangle = \theta_j \ast \theta_j[An] = \delta[n].
\]

Taking the Fourier transform of this equality, and using exercise 3.20, one has that this is equivalent to

\[
A^{-1} \sum_{k=0}^{A-1} |\hat{\theta}_j(A^{-1}(\omega - 2k\pi))|^2 = 1. \tag{9}
\]

With a similar derivation, for \( B = 2^{j+1-L} \), the families \( \{\theta_{j+1}^0[m - Bn]\}_n \) and \( \{\theta_{j+1}^1[m - Bn]\}_n \) are orthogonal sets of vectors orthogonal to each other if and only if

\[
\left\{ \begin{array}{l}
B^{-1} \sum_{k=0}^{B-1} |\hat{\theta}_j^0(B^{-1}(\omega - 2k\pi))\hat{\theta}_j^0(B^{-1}(\omega - 2k\pi))^*| = 0,
B^{-1} \sum_{k=0}^{B-1} |\hat{\theta}_j^1(B^{-1}(\omega - 2k\pi))\hat{\theta}_j^1(B^{-1}(\omega - 2k\pi))^*| = 1, \forall s = 0, 1.
\end{array} \right.
\]

Similarly to the proof of Theorem 8.1, one verifies these relationships using (9) and

\[
\hat{\theta}_{j+1}^0(\omega) = \hat{\theta}_j(\omega)\hat{h}(A\omega) \quad \text{and} \quad \hat{\theta}_{j+1}^1(\omega) = \hat{\theta}_j(\omega)\hat{g}(A\omega).
\]

Exercise 8.3. Each node in the tree has \( 2^p \) children. Fixing two quadrature mirror filters \((h_0, h_1)\), for each \( 0 \leq s < 2^p \), one defines the filters

\[
h_s[n] = \prod_{i=0}^{p-1} h^{s_i}[n_i]
\]

where \((s_i)_i\) is the binary expansion of \( s \). The computation of the wavelet packet coefficients is performed by the following filtering process from the top to the bottom of the tree

\[
d_{j+1}^{2^p s + [n]} = d_j^s \ast h_{s}[2n].
\]

The number of scale is \( \log_2(N)/d \), the number of nodes per scale is \( 2^pj \), the number of coefficients per node is \( N/2^pj \), so that the number of coefficients per scale is \( N \). The complexity per scale is \( O(KN) \) where \( K \) is the length of the filter, so that the overall complexity is \( O(KN \log(N)) \).

Exercise 8.4. There are \( \sqrt{N} \log(N) \) horizontal and vertical atoms, so \( N \log(N)^2 \) atoms in total.

One applies the wavelet packet decomposition algorithm along each of the \( \sqrt{N} \) rows in \( O(K\sqrt{N} \log(\sqrt{N})) \) operations per row. Then one applies the same process to the column. The overall complexity is thus \( O(KN \log(N)) \) operations.
**Exercise 8.5.** a) Denoting \( \alpha = (1 - i)/2 \), one has

\[
g_k = \alpha e_k + \bar{\alpha} e_k \quad \text{where} \quad e_k[m] = e^{\frac{2\pi i}{N} km}.
\]

Thus one has

\[
\langle g_k, g_\ell \rangle = \langle \alpha e_k, \alpha e_\ell \rangle + \langle \alpha e_k, \bar{\alpha} e_\ell \rangle + \langle \bar{\alpha} e_k, \alpha e_\ell \rangle + \langle \bar{\alpha} e_k, \bar{\alpha} e_\ell \rangle
\]

One conclude by noticing that \( 2|\alpha|^2 = N \) and \( \alpha^2 = -\bar{\alpha}^2 \).

b) One has

\[
\langle f, g_k \rangle = \bar{\alpha} \langle f, e_k \rangle + \alpha \langle f, e_k \rangle = \bar{\alpha} \hat{f}[k] + \alpha \hat{f}[k]
\]

**Exercise 8.6.** One considers the following 2-periodic function

\[
\tilde{f}(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t < 1, \\
  -f(-t) & \text{if } -1 \leq t < 0, \\
  f(2-t) & \text{if } 1 \leq t < 2, \\
  -f(t+2) & \text{if } -2 \leq t < -1.
\end{cases}
\]

Since \( \tilde{f} \) is antisymmetric, one can expand orthogonally

\[
\tilde{f}(t) = \sum_k b_k \sin(\pi kt/2)
\]

and the other symmetries of \( \tilde{f} \) with respect to 1 and \(-1\) implies that \( b_{2k} = 0 \).

Since

\[
b_{2k+1} = \langle \tilde{f}, \sin(\pi (k + 1/2)t) \rangle = 2 \langle \tilde{f}, \sin(\pi (k + 1/2)t) \rangle
\]

one sees that \( \left\{ \sqrt{2} \sin(\pi (k + 1/2)t) \right\}_k \) is an orthonormal basis of \( L^2[0, 1] \).

For a discrete signal \( f \in \mathbb{C}^N \), performing anti-symmetry about \(-1/2\) and symmetry about \( N - 1/2 \) and \(-N + 1/2\) shows that

\[
g_k[n] = \sqrt{\frac{2}{N}} \sin\left(\frac{\pi}{N}(k + 1/2)(n + 1/2)\right)
\]

is an orthogonal basis of \( \mathbb{C}^N \).

**Exercise 8.7.** One considers the following 2-periodic function

\[
\tilde{f}(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t < 1, \\
  -f(-t) & \text{if } -1 \leq t < 0.
\end{cases}
\]

Since \( \tilde{f} \) is antisymmetric, one can expand orthogonally

\[
\tilde{f}(t) = \sum_k b_k \sin(\pi kt)
\]

and since

\[
b_k = \langle \tilde{f}, \sin(\pi kt) \rangle = 2 \langle \tilde{f}, \sin(\pi kt) \rangle
\]

one sees that \( \left\{ \sqrt{2} \sin(\pi kt) \right\}_k \) is an orthonormal basis of \( L^2[0, 1] \).
For a discrete signal \( f \in \mathbb{C}^N \), performing anti-symmetry about \(-1/2\) shows that
\[
g_k[n] = \sqrt{\frac{2}{N}} \sin \left( \frac{\pi}{N} k(n + 1/2) \right)
\]
is an orthogonal basis of \( \mathbb{C}^N \).

**Exercise 8.9.** The Meyer wavelet \( \psi \) is written \( \hat{\psi}(\omega) = g(\omega) e^{i\pi \omega} \). A Meyer wavelet \( \psi_{j,n}(x) = 2^{-j/2}g(x/2^j - n) \) satisfies
\[
\hat{\psi}_{j,n}(\omega) = 2^{j/2}b(2^j \omega) \left( \cos(2\pi(k/1/2)2^j \omega) - i \sin(2\pi(k/1/2)2^j \omega) \right)
\]
for a windowing function that verifies some compatibility conditions, so that
\[
\langle f, \psi_{j,n} \rangle = \langle \hat{f}, \hat{\psi}_{j,n} \rangle = 2^{j/2} \langle \hat{f}, g_{1,j,k} \rangle - i 2^{j/2} \langle \hat{f}, g_{2,j,k} \rangle
\]
where \( \{g_{1,j,k}\}_{j,k} \) is a lapped cosine basis and \( \{g_{2,j,k}\}_{j,k} \) is a lapped sine basis.

The Meyer wavelet coefficients can be computed by applying an \( O(N \log(N)) \) FFT followed by \( \log(N) \) lapped transform, and thus the complexity of this algorithm is \( O(N \log(N)^2) \). The computation of the filter bank wavelet algorithm with FFT would take \( O(N \log(N)) \) operation, but would not provide as accurate results.

**Exercise 8.12.**

a) An admissible tree is a an admissible binary tree (called root tree) with a collection of admissible binary trees indexed by the leafs of the tree (called leaf trees).

b) We denote as \( C_{j,k} \) the number of admissible double trees with a root tree of depth at most \( j \) and with leaf trees of depth at most \( k \). One has
\[
C_{j,j} = C_{j-1,j}^2 + 1 \geq C_{j-1,j}^2
\]
and \( C_{0,j} = B_j \) where \( B_k \) is the number of admissible binary tree, so that
\[
C_{j,j} \geq (B_j)^{2^{j-1}} \geq 2^{(j-1)2^{j-1}}.
\]

Similarly to the proof of Theorem 8.2,
\[
\log_2(C_{j+1,j+1}) \leq 2 \log(C_{j,j+1}) + 1/4 \leq 2^j \log_2(C_{1,j+1}) + \frac{1}{4} \sum_{i=0}^{j-1} 2^i.
\]

Since
\[
C_{1,j+1} = B_{j+1} + B_{j+1}^2 \leq 2 \cdot 2^j + 2 \cdot 2^j,
\]
on one obtains
\[
\log_2(C_{j+1,j+1}) \leq 2^j \log_2(2 \cdot 2^j + 2 \cdot 2^j) + \frac{1}{4} 2^j.
\]

8 Chapter 9

**Exercise 9.1.** a) One has
\[
|f - f_N|^2 = \sum_{m \geq N} |\langle f, g_m \rangle|^2.
\]

Using the frame property, and denoting \((A, B)\) the frame bounds, one has
\[
A \sum_{m \geq N} |\langle f, g_m \rangle|^2 \leq |f - f_N|^2 \leq B \sum_{m \geq N} |\langle f, g_m \rangle|^2.
\]
AC \leq \sum_{N=0}^{\infty} N^{2s-1} \varepsilon_1(N, f) \leq BC \quad \text{where} \quad C = \sum_{m \geq 0} |\langle f, g_m \rangle|^2 \sum_{N \geq 0} N^{2s-1}.

The following is similar to the proof of Theorem 9.1.

b) We denote

\[ \tilde{f}_M = \sum_{k < M} \langle f, g_{m_k}\rangle \tilde{g}_{m_k}, \]

and \( f_M \) the best \( M \) term approximation. One has, using the frame property

\[ \|f - f_M\|^2 \leq \|f - \tilde{f}_M\|^2 = \sum_{k \geq M} |\langle f, g_{m_k}\rangle|^2 \tilde{g}_{m_k}^2 \leq B \sum_{k \geq M} |\langle f, g_{m_k}\rangle|^2 = O(\sum_{k \geq M} k^{-2s}) = O(M^{1-2s}). \]

Exercise 9.4. An \( M \)-term approximation of the multi-channel signal is defined as

\[ f_\Lambda = (f_{k,\Lambda})_k \quad \text{where} \quad f_{k,\Lambda} = \sum_{m \in \Lambda} \langle f_k, g_m \rangle g_m, \]

where \( \Lambda \) is a set of \( M = |\Lambda| \) indexes. One has

\[ \|f - f_\Lambda\|^2 = \sum_{m \notin \Lambda} A_m \quad \text{where} \quad A_m = \sum_k |\langle f_k, g_m \rangle|^2. \]

The best support \( \Lambda \) that minimize the approximation error is thus the one that selects the \( M \) largest values of \( A_m \).

Exercise 9.5. One has, for \( N = 2^{-j} \)

\[ f_N(t) = P_{\sqrt{2}}(f)(t) = f \star \Phi(Nt) \]

where \( \Phi \) is a smooth low frequency function.

We assume that \( \Phi \) is smooth and compactly supported in \([-K/2, K/2]\), and that \( \int \Phi = 1, |\Phi|_\infty \leq C \). One then checks that the approximation error is localized in \([-K/2, K/2]/N\)

\[ \int_0^1 |f(t) - f \star \Phi(Nt)|^2 dt \leq 2CKN^{-1}. \]

Exercise 9.6. a) For a smooth function, one has

\[ |f(x)| = | \int_0^x f'(x) dx + f(0) | \leq \int_0^x |f'(x)| dx + |f(0)| \leq TV(f) + |f(0)| < +\infty. \]

This result carries over to arbitrary function by approximating by a smooth function.

b) We consider \( f(x) = |x|^{-\alpha} \phi(x) \) where \( \phi \) is some smooth localizing function, that is 1 inside the disc of radius 1. One has, for \( |x| \leq 1 \)

\[ \nabla f(x) = -\alpha |x|^{x} \phi(x). \]
Using a polar change of variables,
\[
\int |\nabla f(x)|\,dx = C + \alpha \int \frac{dx}{|x|^{\alpha+1}} = C + 2\pi \alpha \int_0^1 \frac{dx}{r^\alpha}
\]
where \(C\) account for the total variation outside the unit disk. One can see that for \(0 < \alpha < 1\), \(f\) is of bounded variation, but \(f\) is not bounded.

**Exercise 9.7.** Formally, we define
\[
g(t) = \int_\mathbb{R} H(\omega, t)\,d\omega \quad \text{where} \quad H(\omega, t) = (i\omega)^p \hat{f}(\omega)e^{i\omega t}.
\]
One has
\[
|H(\omega, t)| \leq |\omega|^p |\hat{f}(\omega)|
\]
Using Cauchy-Schartz inequality, one has
\[
\int_{|\omega| > \eta} |\omega|^p |\hat{f}(\omega)|\,d\omega \leq \left( \int_{|\omega| > \eta} |\omega|^{2(p-s)}\,d\omega \right)^{1/2} \|f\|_{Sob(s)}
\]
which is bounded if \(s > p + 1/2\). So using classical Theorem of derivation under the sign \(f\), this shows that \(f\) is \(C^\alpha\) and \(f^{(\alpha)} = g\).

**Exercise 9.8.** One can verify numerically that Fourier and orthogonal polynomial approximations behave similarly. An orthogonal polynomial of degree \(p\) has \(p\) oscillations and is similar to a sinusoid function. The linear \(N\)-term approximation with orthogonal polynomial of a \(C^\alpha\) function is better that the error produced by a Taylor approximation, and is thus \(\|f - f_N\|^2 = O(N^{-2\alpha})\)

**Exercise 9.9.** One has
\[
\|f - f_M\|^2 \leq \sum_{m=0}^M (t_{k+1} - t_k)\Delta = \Delta.
\]
We use the fact that any bounded variation function can be decomposed as \(f = f_1 - f_2\) where \(f_1, f_2\) are increasing functions. Without loss of generality, we assume \(f_1(0) = f_2(0) = 0\) and \(f_1(1) = f_2(1) = 1\).

For each function \(f_i\), we build a set \(T_i = \{t_k^i = f_i^{-1}(i/M)\}_{k=0}^{M-1}\) of points which guarantees that \(f_i\) varies of at most \(\Delta = 1/M\) over each \([t_k^i, t_{k+1}^i]\). Defining \(\{t_k\}_{k=0}^{2M-1} = T_1 \cup T_2\) guarantees that \(f\) varies of less than \(\Delta = 1/M\) over each \([t_k, t_{k+1}]\). One thus has \(\|f - f_M\|^2 = O(M^{-1})\).

**Exercise 9.11.** We consider a function \(f\) such that \(\|f, \psi_{j,n}\| = 2^n\). For dyadic \(N = 2^J\), and ordering wavelet coefficients from coarse to fine scales, \(\varepsilon_n[N] = \varepsilon_l[N]\).

For \(\alpha < s - 1/2\), one has
\[
\sum_{j \leq 0} \sum_n |\langle f, \psi_{j,n} \rangle|^2 2^{-2j\alpha} = \sum_j 2^{-j} 2^{-2\alpha j} 2^{2j} = \sum_{j \leq 0} 2^{2j(s-\alpha-1/2)} < +\infty
\]
and Theorem 9.4 shows that \(f\) is in \(W^\alpha\).

**Exercise 9.12.** b) Denoting \(|\text{supp}(\psi)| = C = 8\), one has for the \(CK\) wavelets whose support intersect a singularity at scale \(2^J\)
\[
|\langle f, \psi_{j,n} \rangle| \leq |f|_{\infty} |\psi|_{2J}^{1/2},
\]
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the other coefficients being zero. One has
\[ \varepsilon_n[M] \leq 2CK\|f\|_{\infty}^2\|\psi\|_{\infty}^2 M^{-1}. \]
Since there is only \( KC \) non-zero coefficients per scale, the non-linear approximation selects all non-zero coefficients corresponding to wavelets whose support intersects a singularity at scale \( 2^j \) up to scale \( -j = M/(KC) \)
\[ \varepsilon_n[M] \leq \sum_{j \leq J} CK\|f\|_{\infty}^2\|\psi\|_{\infty}^2 2^j \leq 2CK\|f\|_{\infty}^2\|\psi\|_{\infty}^2 = 2CK\|f\|_{\infty}^2\|\psi\|_{1}^2 \omega^{-M}, \]
where \( \omega = 2^{-1/(KC)} \).

Exercise 9.13. a) One has
\[ \arg\min_{a \in \mathbb{R}} \sum_{n=\ell}^{k} |f[n] - a|^2 = \frac{1}{\ell-k+1} \sum_{n=\ell}^{k} f[n]. \]

b) We call \( V_{p,\ell} \) the set of signals supported on \([0, \ldots, \ell]\), that assumes less than \( p \) different values. Any signal \( f_k \in V_{p,\ell} \) can be decomposed as
\[ f_k = f_{\ell} + a_{1}\epsilon_{\ell,k} \]
where \( f_{\ell} \in V_{p-1,\ell-1} \) and \( a \in \mathbb{R} \). One thus has
\[ \min_{f_k \in V_{p,\ell}} \| f - f_k\|_{[0,\ell]} = \min_{f_{\ell} \in [0,\ell-1]} \min_{f \in V_{p-1,\ell-1}} \| f - f_{\ell}\|_{[0,\ell-1]} + \min_{a \in \mathbb{R}} \| f - a\|_{[\ell,k]}, \]
which gives the result.

An algorithm can computes \( \epsilon_{p,k} \) for increasing \( p \) and \( k \), and \( \Sigma_{p,k} \) which are the locations of discontinuities in the optimal signal in \( V_{p,k} \).
One initializes:
- For all \( k \), \( \epsilon_{1,k} = c_{1,k} \) and \( \Sigma_{1,k} = \emptyset \).
- For all \( p \), \( \epsilon_{p,1} = 0 \) and \( \Sigma_{p,0} = \{0, \ldots, 0\} \) (\( p \) times).
For \( p = 2, \ldots, K \), for \( k = 1, \ldots, N-1 \), compute
\[ \epsilon_{p,k} = \min_{\ell \in [0,k-1]} \epsilon_{p-1,\ell} + c_{\ell,k} \]
and denoting \( \ell^* \) such that \( \epsilon_{p,k} = \epsilon_{p-1,\ell^*} + c_{\ell^*,k} \), update
\[ \Sigma_{p,k} = \Sigma_{p-1,\ell^*} \cup \{\ell^*\}. \]
After running this algorithm, \( f_{K,N} \) has discontinuities in \( \Sigma_{K,N} \). The main numerical cost is the computation of the \( c_{p,k} \), which takes \( O(N^2) \) for each \( k \), so the overall complexity is \( O(KN^2) \).

Exercise 9.14. a) Performing a first order approximation of the phase near point \( 2^j n \), and assuming that the amplitude is nearly constant over the support of size \( 2^j \), one gets
\[ f(t) \approx a(2^j n) \exp(i\phi(2^j n) + i\phi'(2^j n)(t - 2^j n)). \]
One thus has
\[ |\langle f, \psi_{j,n} \rangle| \approx |a(2^j n)2^{-j/2}(\psi(t/2^j - n), \exp(i\phi'(2^j n)(t - 2^j n)))| \]
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and thus performing the change of variable $t/2^j - n \mapsto t/2^j - n$ in the inner product, one obtains

$$|\langle f, \psi_{j,n} \rangle| \approx a(2^j n)^{2j/2} |\hat{\psi}(2^j \phi'(2^j n))|.$$

b) For $f(t) = \sin(1/t)$, one has $\phi(t) = 1/t$, $a = 1$, $\phi'(t) = -1/t^2$, and thus

$$|\langle f, \psi_{j,n} \rangle| \approx 2^{j/2} |\psi(2^{-j} n^{-2})|.$$

The $\ell^p$ norm of the coefficients reads

$$\sum_{j \leq 0, 0 \leq n \leq 2^{-j}} |\langle f, \psi_{j,n} \rangle|^p \approx \sum_{j \leq 0, 0 \leq n \leq 2^{-j}} |2^{jp/2} |\hat{\psi}(2^{-j} n^{-2})|^p. \quad (10)$$

The wavelet function is close to a band pass filter (Shannon wavelet), so $|\hat{\psi}(\omega)|$ is nearly constant equal to $A > 0$ in some interval $[C_1, C_2]$, and small outside. As an approximation, we can thus consider that for each $j$ the sum in (10) is well approximated by restricting it to indexes $n$ that satisfies

$$C_1 \leq 2^{-j} n^{-2} \leq C_2 \quad \Rightarrow \quad C_2^{-1/2} 2^{-j/2} \leq n \leq C_1^{-1/2} 2^{-j/2}.$$

For each $j$, the number of elements in the sum (10) is thus approximately $C 2^{-j/2}$ for some constant $C$, so that

$$\sum_{j \leq 0, 0 \leq n \leq 2^{-j}} |\langle f, \psi_{j,n} \rangle|^p \approx CA \sum_{j \leq 0} 2^{j/2(p-1)}. \quad (11)$$

This $\ell^p$ norm is thus finite if and only if $p > 1$.

c) Theorem 9.10 tells us that for all $p < 1$, $\epsilon_n[M] = O(M^{1-2/p})$, so that $\epsilon_n[M] = O(M^{-\alpha})$ for any $\alpha < 1$.  

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